The four-color problem was a main driving force for the development of graph theory as we know it today, and coloring is still a topic that many graph theorists like best. Here is a simple-sounding coloring problem, raised by Jeff Dinitz in 1978, which defied all attacks until its astonishingly simple solution by Fred Galvin fifteen years later.

Consider \( n^2 \) cells arranged in an \((n \times n)\)-square, and let \((i, j)\) denote the cell in row \(i\) and column \(j\). Suppose that for every cell \((i, j)\) we are given a set \(C(i, j)\) of \(n\) colors. Is it then always possible to color the whole array by picking for each cell \((i, j)\) a color from its set \(C(i, j)\) such that the colors in each row and each column are distinct?

As a start consider the case when all color sets \(C(i, j)\) are the same, say \(\{1, 2, \ldots, n\}\). Then the Dinitz problem reduces to the following task: Fill the \((n \times n)\)-square with the numbers \(1, 2, \ldots, n\) in such a way that the numbers in any row and column are distinct. In other words, any such coloring corresponds to a Latin square, as discussed in the previous chapter. So, in this case, the answer to our question is “yes.”

Since this is so easy, why should it be so much harder in the general case when the set \(C := \bigcup_{i,j} C(i, j)\) contains even more than \(n\) colors? The difficulty derives from the fact that not every color of \(C\) is available at each cell. For example, whereas in the Latin square case we can clearly choose an arbitrary permutation of the colors for the first row, this is not so anymore in the general problem. Already the case \(n = 2\) illustrates this difficulty.

Suppose we are given the color sets that are indicated in the figure. If we choose the colors 1 and 2 for the first row, then we are in trouble since we would then have to pick color 3 for both cells in the second row.

Before we tackle the Dinitz problem, let us rephrase the situation in the language of graph theory. As usual we only consider graphs \(G = (V, E)\) without loops and multiple edges. Let \(\chi(G)\) denote the chromatic number of the graph, that is, the smallest number of colors that one can assign to the vertices such that adjacent vertices receive different colors.

In other words, a coloring calls for a partition of \(V\) into classes (colored with the same color) such that there are no edges within a class. Calling a set \(A \subseteq V\) independent if there are no edges within \(A\), we infer that the chromatic number is the smallest number of independent sets which partition the vertex set \(V\).
In 1976 Vizing, and three years later Erdős, Rubin, and Taylor, studied the following coloring variant which leads us straight to the Dinitz problem. Suppose in the graph $G = (V, E)$ we are given a set $C(v)$ of colors for each vertex $v$. A **list coloring** is a coloring $c: V \rightarrow \bigcup_{v \in V} C(v)$ where $c(v) \in C(v)$ for each $v \in V$. The definition of the **list chromatic number** $\chi_l(G)$ should now be clear: It is the smallest number $k$ such for any list of color sets $C(v)$ with $|C(v)| = k$ for all $v \in V$ there always exists a list coloring. Of course, we have $\chi_l(G) \leq |V|$ (we never run out of colors). Since ordinary coloring is just the special case of list coloring when all sets $C(v)$ are equal, we obtain for any graph $G$

$$\chi(G) \leq \chi_l(G).$$

To get back to the Dinitz problem, consider the graph $S_n$ which has as vertex set the $n^2$ cells of our $(n \times n)$-array with two cells adjacent if and only if they are in the same row or column. Since any $n$ cells in a row are pairwise adjacent we need at least $n$ colors. Furthermore, any coloring with $n$ colors corresponds to a Latin square, with the cells occupied by the same number forming a color class. Since Latin squares, as we have seen, exist, we infer $\chi(S_n) = n$, and the Dinitz problem can now be succinctly stated as

$$\chi_l(S_n) = n?$$

One might think that perhaps $\chi(G) = \chi_l(G)$ holds for any graph $G$, but this is a long shot from the truth. Consider the graph $G = K_{2,4}$. The chromatic number is 2 since we may use one color for the two left vertices and the second color for the vertices on the right. But now suppose that we are given the color sets indicated in the figure.

To color the left vertices we have the four possibilities $1|3, 1|4, 2|3$ and $2|4$, but any one of these pairs appears as a color set on the right-hand side, so a list coloring is not possible. Hence $\chi_l(G) \geq 3$, and the reader may find it fun to prove $\chi_l(G) = 3$ (there is no need to try out all possibilities!). Generalizing this example, it is not hard to find graphs $G$ where $\chi_l(G) = 2$, but $\chi_l(G)$ is arbitrarily large! So the list coloring problem is not as easy as it looks at first glance.

Back to the Dinitz problem. A significant step towards the solution was made by Jeanette Janssen in 1992 when she proved $\chi_l(S_n) \leq n + 1$, and the **coup de grâce** was delivered by Fred Galvin by ingeniously combining two results, both of which had long been known. We are going to discuss these two results and show then how they imply $\chi_l(S_n) = n$.

First we fix some notation. Suppose $v$ is a vertex of the graph $G$, then we denote as before by $d(v)$ the **degree** of $v$. In our square graph $S_n$ every vertex has degree $2n - 2$, accounting for the $n - 1$ other vertices in the same row and in the same column. For a subset $A \subseteq V$ we denote by $G_A$ the subgraph which has $A$ as vertex set and which contains all edges of $G$ between vertices of $A$. We call $G_A$ the subgraph induced by $A$, and say that $H$ is an **induced subgraph** of $G$ if $H = G_A$ for some $A$. 

The graph $S_3$