Here is an appealing problem which was raised by Victor Klee in 1973. Suppose the manager of a museum wants to make sure that at all times every point of the museum is watched by a guard. The guards are stationed at fixed posts, but they are able to turn around. How many guards are needed?

We picture the walls of the museum as a polygon consisting of \( n \) sides. Of course, if the polygon is convex, then one guard is enough. In fact, the guard may be stationed at any point of the museum. But, in general, the walls of the museum may have the shape of any closed polygon.

Consider a comb-shaped museum with \( n = 3m \) walls, as depicted on the right. It is easy to see that this requires at least \( m = \frac{n}{3} \) guards. In fact, there are \( n \) walls. Now notice that the point 1 can only be observed by a guard stationed in the shaded triangle containing 1, and similarly for the other points 2, 3, \ldots, \( m \). Since all these triangles are disjoint we conclude that at least \( m \) guards are needed. But \( m \) guards are also enough, since they can be placed at the top lines of the triangles. By cutting off one or two walls at the end, we conclude that for any \( n \) there is an \( n \)-walled museum which requires \( \left\lfloor \frac{n}{3} \right\rfloor \) guards.
The following result states that this is the worst case.

**Theorem.** For any museum with $n$ walls, $\left\lceil \frac{n}{3} \right\rceil$ guards suffice.

This “art gallery theorem” was first proved by Vašek Chvátal by a clever argument, but here is a proof due to Steve Fisk that is truly beautiful.

**Proof.** First of all, let us draw $n - 3$ noncrossing diagonals between corners of the walls until the interior is triangulated. For example, we can draw 9 diagonals in the museum depicted in the margin to produce a triangulation. It does not matter which triangulation we choose, any one will do. Now think of the new figure as a plane graph with the corners as vertices and the walls and diagonals as edges.

**Claim.** This graph is 3-colorable.

For $n = 3$ there is nothing to prove. Now for $n > 3$ pick any two vertices $u$ and $v$ which are connected by a diagonal. This diagonal will split the graph into two smaller triangulated graphs both containing the edge $uv$. By induction we may color each part with 3 colors where we may choose color 1 for $u$ and color 2 for $v$ in each coloring. Pasting the colorings together yields a 3-coloring of the whole graph.

The rest is easy. Since there are $n$ vertices, at least one of the color classes, say the vertices colored 1, contains at most $\left\lfloor \frac{n}{3} \right\rfloor$ vertices, and this is where we place the guards. Since every triangle contains a vertex of color 1 we infer that every triangle is guarded, and hence so is the whole museum. □

The astute reader may have noticed a subtle point in our reasoning. Does a triangulation always exist? Probably everybody’s first reaction is: Obviously, yes! Well, it does exist, but this is not completely obvious, and, in fact, the natural generalization to three dimensions (partitioning into tetrahedra) is false! This may be seen from Schönhardt’s polyhedron, depicted on the left. It is obtained from a triangular prism by rotating the top triangle, so that each of the quadrilateral faces breaks into two triangles with a nonconvex edge. Try to triangulate this polyhedron! You will notice that any tetrahedron that contains the bottom triangle must contain one of the three top vertices: but the resulting tetrahedron will not be contained in Schönhardt’s polyhedron. So there is no triangulation without an additional vertex.

To prove that a triangulation exists in the case of a planar nonconvex polygon, we proceed by induction on the number $n$ of vertices. For $n = 3$ the polygon is a triangle, and there is nothing to prove. Let $n \geq 4$. To use induction, all we have to produce is one diagonal which will split the polygon $P$ into two smaller parts, such that a triangulation of the polygon can be pasted together from triangulations of the parts.

Call a vertex $A$ *convex* if the interior angle at the vertex is less than $180^\circ$. Since the sum of the interior angles of $P$ is $(n - 2)180^\circ$, there must be a