An Improved Deterministic #SAT Algorithm for Small De Morgan Formulas

Ruiwen Chen¹, Valentine Kabanets¹, and Nitin Saurabh²

¹ Simon Fraser University, Burnaby, Canada
ruiwenc@sfu.ca, kabanets@cs.sfu.ca
² Institute of Mathematical Sciences, Chennai, India
nitin@imsc.res.in

Abstract. We give a deterministic #SAT algorithm for de Morgan formulas of size up to \(n^{2.63}\), which runs in time \(2^{n-O(1)}\). This improves upon the deterministic #SAT algorithm of [3], which has similar running time but works only for formulas of size less than \(n^{2.5}\).

Our new algorithm is based on the shrinkage of de Morgan formulas under random restrictions, shown by Paterson and Zwick [12]. We prove a concentrated and constructive version of their shrinkage result. Namely, we give a deterministic polynomial-time algorithm that selects variables in a given de Morgan formula so that, with high probability over the random assignments to the chosen variables, the original formula shrinks in size, when simplified using a deterministic polynomial-time formula-simplification algorithm.

Keywords: de Morgan formulas, random restrictions, shrinkage, SAT algorithms.

1 Introduction

Subbotovskaya [16] introduced the method of random restrictions to prove that Parity requires de Morgan formulas of size \(Ω(n^{1.5})\), where a de Morgan formula is a boolean formula over the basis \(\{\lor, \land, \neg\}\). She showed that a random restriction of all but a fraction \(p\) of the input variables yields a new formula whose size is expected to reduce by at least the factor \(p^{1.5}\). That is, the shrinkage exponent \(Γ\) for de Morgan formulas is at least 1.5, where the shrinkage exponent is defined as the least upper bound on \(γ\) such that the expected formula size shrinks by the factor \(p^γ\) under a random restriction leaving \(p\) fraction of variables free.

Impagliazzo and Nisan [9] argued that Subbotovskaya’s bound \(Γ ≥ 1.5\) is not optimal, by showing that \(Γ ≥ 1.556\). Paterson and Zwick [12] improved upon [9], getting \(Γ ≥ (5−\sqrt{3})/2 ≈ 1.63\). Finally, Håstad [6] proved the tight bound \(Γ = 2\); combined with Andreev’s construction [1], this yields a function in \(P\) requiring de Morgan formulas of size \(Ω(n^{3-o(1)})\).

While the original motivation to study shrinkage in [16,9,12,6] was to prove formula lower bounds, the same results turn out to be useful also for designing non-trivial SAT algorithms for small de Morgan formulas. Santhanam [14] strengthened
Subbotovskaya’s expected shrinkage result to concentrated shrinkage, i.e., shrinkage with high probability, and used this to get a deterministic #SAT algorithm (counting the number of satisfying assignments) for linear-size de Morgan formulas, with the running time $2^{n - \Omega(n)}$. Santhanam’s algorithm deterministically selects a most frequent variable in the current formula, and recurses on the two subformulas obtained by restricting the chosen variable to 0 and 1; after $n - \Omega(n)$ recursive calls, almost all obtained formulas depend on fewer than the actual number of free variables remaining, which leads to nontrivial savings over the brute-force SAT algorithm for the original formula. A similar algorithm works also for formulas of size less than $n^{2.5}$, with the running time $2^{n - n^{\Omega(1)}}$.

Motivated by average-case formula lower bounds, Komargodski et al. [11] (building upon [8]) showed a concentrated-shrinkage version of Håstad’s optimal result for the shrinkage exponent $\Gamma = 2$. Combined with the aforementioned algorithm of Chen et al. [3], this yields a nontrivial randomized zero-error #SAT algorithm for de Morgan formulas of size $n^{3 - o(1)}$, running in time $2^{n - n^{\Omega(1)}}$.

The main question addressed by our paper is whether there is a deterministic #SAT algorithm, with similar running time, for formulas of size close to $n^3$. This question is interesting since getting a deterministic algorithm often yields deeper understanding of the problem by revealing additional structural properties. It also provides better understanding of the role of randomness in efficient algorithms, as part of research on derandomization.

We give a deterministic #SAT algorithm for formulas of size up to $n^{2.63}$. In the process, we refine the results of Paterson and Zwick [12] on shrinkage of de Morgan formulas by making their results constructive in a certain precise sense. We provide more details next.

1.1 Our Main Results and Techniques

Our main result is a deterministic #SAT algorithm for de Morgan formulas of size up to $n^{2.63}$, running in time $2^{n - n^{\Omega(1)}}$.

**Theorem 1 (Main).** There is a deterministic algorithm for counting the number of satisfying assignments in a given de Morgan formula on $n$ variables of size at most $n^{2.63}$ which runs in time at most $2^{n - n^{\delta}}$, for some constant $0 < \delta < 1$.

As in [14, 3], we use a deterministic algorithm to choose a next variable to restrict, and then recurse on the two resulting restrictions of this variable to 0 and 1. Instead of Subbotovskaya-inspired selection procedure (choosing the most frequent variable), we use the weight function introduced by Paterson and Zwick [12], which measures the potential savings for each one-variable restriction, and selects a variable with the biggest savings. Since [12] gives the shrinkage exponent $\Gamma \approx 1.63$, rather than Subbotovskaya’s 1.5, this could potentially lead to an improved #SAT algorithm for larger de Morgan formulas.

However, computing the savings, as defined by [12], is NP-hard, as it requires computing the size of a smallest logical formula equivalent to a given one-variable restriction. In fact, the shrinkage result of [12] is nonconstructive in the following