Chapter 23
Invariant Families of Manifolds

If \( x_1, \ldots, x_n \) are point coordinates of an \( n \)-times extended space, then a family of manifolds of this space is represented by equations of the form:

\[
\Omega_1(x_1, \ldots, x_n, l_1, \ldots, l_m) = 0, \ldots, \Omega_{n-q}(x_1, \ldots, x_n, l_1, \ldots, l_m) = 0,
\]

in which, aside from the variables \( x_1, \ldots, x_n \), certain parameters: \( l_1, \ldots, l_m \) are present.

If one executes an arbitrary transformation:

\[
x_i' = f_i(x_1, \ldots, x_n)
\]

of the space \( x_1, \ldots, x_n \), then each one of the manifolds (1) is transferred to a new manifold, hence the whole family (1) converts into a new family of manifolds. One obtains the equations of this new family when one takes away \( x_1, \ldots, x_n \) from (1) with the help of the \( n \) equations: \( x_i' = f_i(x) \). Now in particular, if the new family of manifolds coincides with the original family (1), whence every manifold of the family (1) is transferred by the transformation: \( x_i' = f_i(x) \) to a manifold of the same family, then we say that the family (1) admits the transformation in question, or that it remains invariant by it.

If a family of manifolds in the space \( x_1, \ldots, x_n \) admits all transformations of an \( r \)-term group, then we say that it admits the group in question.

Examples of invariant families of manifolds in the space \( x_1, \ldots, x_n \) have already appeared to us several times; every intransitive group decomposes the space into an invariant family of individually invariant manifolds (Chap. 13, pp. 227–228); every imprimitive group decomposes the space into an invariant family of manifolds that it permutes (p. 232 sq.); also, every manifold individually invariant by a group may be interpreted as an invariant family of manifolds, namely as a family parametrized by a point.

In what follows, we now consider a completely arbitrary family of manifolds. First, we study under which conditions this family admits a single given transformation, or a given \( r \)-term group. Then, we imagine that a group is given by which the family remains invariant and we determine the law according to which the manifolds
of the family are permuted with each other by the transformations of this group. In this way, we obtain a new process to set up the groups which are isomorphic with a given group. Finally, we give a method for finding all families of manifolds invariant by a given group.

§ 110. Let the equations:

\[ \Omega_k(x_1, \ldots, x_n, l_1, \ldots, l_m) = 0 \quad (k = 1 \cdots n-q) \]

with the \( m \) arbitrary parameters \( l_1, \ldots, l_m \) represent an arbitrary family of manifolds.

If \( l_1, \ldots, l_m \) are absolutely arbitrary parameters, then naturally, one should not be able to eliminate all the \( x \) from (1); therefore, the equations (1) must be solvable with respect to \( n-q \) of the variables \( x_1, \ldots, x_n \).

By contrast, it is not excluded that the \( l \) can be eliminated from (1), and this says nothing but that some relations between the \( x \) alone can be derived from (1). Only the number of independent equations free of the \( l \) which follow from (1) must be smaller than \( n-q \), since otherwise, the parameters \( l_1, \ldots, l_r \) would be only apparent and the equations (1) would therefore represent not a family of manifolds, but a single manifold.

Before we study how the family of manifolds (1) behaves relative to transformations of the \( x \), we must first make a few remarks about the nature of the equations (1).

The equations (1) contain \( m \) parameters \( l_1, \ldots, l_m \); if we let these parameters take all possible values, then we obtain \( \infty^m \) different systems of values: \( l_1, \ldots, l_m \), but not necessarily \( \infty^m \) different manifolds. It must therefore be determined under which conditions the given system of equations represents exactly \( \infty^m \) different manifolds, or in other words: one must give a criterion to determine whether the parameters: \( l_1, \ldots, l_m \) in the equations (1) are essential, or not.

In order to find such a criterion, we imagine that the equations: \( \Omega_k = 0 \) are solved with respect to \( n-q \) of the \( x \), say with respect to \( x_{q+1}, \ldots, x_n \):

\[ x_{q+k} = \psi_{q+k}(x_1, \ldots, x_q, l_1, \ldots, l_m) \quad (k = 1 \cdots n-q) \]

and we imagine moreover that the functions \( \psi_{q+k} \) are expanded with respect to the powers of: \( x_1 - x_1^0, \ldots, x_q - x_q^0 \) in the neighborhood of an arbitrary system of values: \( x_1^0, \ldots, x_q^0 \). The coefficients of the expansion, whose number is naturally infinitely large in general, will be analytic functions of \( l_1, \ldots, l_m \) and we may call them:

\[ \Lambda_j(l_1, \ldots, l_m) \quad (j = 1, 2 \ldots). \]

The question amounts just to how many independent functions are extant amongst all the functions \( \Lambda_1, \Lambda_2, \ldots \).

Indeed, if amongst the \( \Lambda \), there are exactly \( l \) that are mutually independent functions — there are anyway surely no more than \( l \) —, then to the \( \infty^m \) different systems