In this chapter, P/T-nets are given a concurrent semantics by replacing the usual firing rule with the step firing rule, which allows steps $M[e]M'$ where $e$ is a multiset of transitions. Concurrent reachability graphs of P/T-nets are a particular case of Mukund's step transition systems. After reformulating Mukund’s definition of regions in step transition systems, we examine the P/T-net realization problem for step transition systems and for step languages. As an aside, we present two extended forms of the P/T-net realization problem for step transition systems, and an alternative form of the P/T-net realization problem for partially ordered languages.

**Step Firing and the Concurrent Reachability Graph**

Let $N = (P, T, F, M_0)$ be a Petri net system with weighted flow relation $F : (P \times T) \cup (T \times P) \to \mathbb{N}$. Let $E = T \to \mathbb{N}$, thus $E$ is the additive monoid of multisets over $T$, with pointwise addition of multisets and with the null multiset $\mathbf{0}$ as neutral element. A multiset $e \in E$ is called a step. Under the step firing rule, the markings of $N$ evolve by steps $M[e]M'$ where $e \in E$ and the following relations hold for every place $p \in P$:

$$M(p) \geq \sum_{t \in T} e(t) \times F(p, t),$$

$$M'(p) = M(p) + \sum_{t \in T} e(t) \times (F(t, p) - F(p, t)).$$

In particular, $M[\mathbf{0}]M$ for every marking $M$. It is worth noting that for any step $M[e]M'$, if $e = e_1 + e_2$, then there exists an intermediate marking $M_1$ such that $M[e_1]M_1$ and $M_1[e_2]M'$. From this interpolation property and the fact that minimal non-empty steps comprise exactly one occurrence of one transition $t \in T$, it follows that the set of markings that can be reached from $M_0$ under the step firing rule is the set of markings that can be reached from $M_0$ under the standard firing rule, i.e. the reachability set $RS(N)$. 
The concurrent reachability graph of $N$, as opposed to its sequential reachability graph $RG(N)$, is the initialized transition system $CRG(N) = (RS(N), E, \delta, M_0)$ where $E = T \rightarrow N$ and $\delta : RS(N) \times E \rightarrow RS(N)$ is the partial transition map defined as follows: for all $e \in E$ and $M, M' \in RS(N)$, $\delta(M, e) = M'$ if and only if $M[e]M'$.

Concurrent reachability graphs of Petri nets are a particular case of Mukund’s step transition systems defined in [112]. A step transition system over $T$ is an initialized and reachable transition system $A = (S, E, \delta, s_0)$ over $E = T \rightarrow N$ such that $\delta(s, 0) = s$ for all $s \in S$.

Reformulation with $\tau_{PT}$-nets

Given a net $N = (P, T, F, M_0)$ with flow relation $F : (P \times T) \cup (T \times P) \rightarrow N$ as above, let $F' : (P \times T) \rightarrow (N \times N)$ be defined by $F'(p, t) = (F(p, t), F(t, p))$. Then the net $N' = (P, T, F', M_0)$ is a $\tau_{PT}$-net, where $\tau_{PT}$ is the type of general P/T-nets (Example 4.14). Recall that $\tau_{PT} = (N, N \times N, d)$ where the partial transition map $d(n, (i, j))$ is equal to $n - i + j$ if $n - i \geq 0$ and is undefined otherwise. Now $N \times N$ is an additive monoid, with componentwise addition and with $0 = (0, 0)$ as a neutral element. Moreover, this monoid is isomorphic to $\{(1, 0), (0, 1)\} \rightarrow N$. Therefore, $\tau_{PT}$ is not only a transition system but also a step transition system. For every place $p \in P$ and for every step $e \in E$, let $F'(p, e) = \sum_{t \in T} e(t) \times F'(p, t)$, that is $F'(p, e) = (\sum_{t \in T} e(t) \times F(p, t), \sum_{t \in T} e(t) \times F(t, p))$. Then for any marking $M \in RS(N)$ and for any step $e \in E$, the following relation holds:

$$(M[e]M' \in RS(N)) \iff (\forall p \in P) \ M(p) \xrightarrow{F'(p, e)} M'(p) \text{ in } \tau_{PT}.$$ 

Therefore, the concurrent reachability graph $CRG(N)$ of $N$ coincides with the step transition system generated by the $\tau_{PT}$-net $N'$ (Definition 5.24).

9.1 Regions of Step Transition Systems

Step Regions

Let $A = (S, E, \delta, s_0)$ be a step transition system over the monoid $E = T \rightarrow N$ (thus $\delta(s, 0) = s$ for all $s \in S$). According to [112], a region of $A$, or step region, is a pair of maps $r = (r_S, r_T)$, where $r_S : S \rightarrow N$ and $r_T : T \rightarrow N \times N$, such that if we let $r_T(t) = (r^o_t, t^o_r)$, then the following relations hold:

1. $\delta(s, e) \text{ defined } \Rightarrow r_S(s) \geq \sum_{t \in T} e(t) \times (r^o_t)$,
2. $\delta(s, e) = s' \Rightarrow r_S(s') = r_S(s) + \sum_{t \in T} e(t) \times (t^o_r - r^o_t)$.

In the particular case where $A = CRG(N)$ is the concurrent reachability graph of a P/T-net, every place $p$ of $N$ induces a corresponding region of $A$, determined by $r_S(M_0) = M_0(p)$ (where $M_0$ is the initial marking of $N$) and $r_T(t) = (p^o_t, t^o_p)$ for every transition $t \in T$. 
