Chapter 19
Dynamics of Mechanisms

For numerical investigations into the dynamics of mechanisms nonlinear differential equations must be formulated. The formulation given in this chapter allows writing a general-purpose software tool applicable to arbitrary mechanisms. The change from one mechanism to another is accomplished by changing readily available input data. More information see in Wittenburg [4].

19.1 Conservative Single-Degree-of-Freedom Mechanisms

Subject of this introductory section is the following rather special yet frequently arising problem of dynamics. Imagine an arbitrary single-degree-of-freedom mechanism with an input shaft and an output shaft. Both shafts are rotating about frame-fixed axes. Simple examples: Hooke’s joint, the planar four-bar, the spherical four-bar and the Bennett mechanism. The transmission ratio of a single-degree-of-freedom mechanism is the ratio \( i = \dot{\varphi}_1/\dot{\varphi}_2 \) (input angular velocity / output angular velocity). For the four illustrative examples the transmission ratio is known as function of the input angle \( \varphi_1 \). For Hooke’s joint the function is (see (13.7)) \( i(\varphi_1) = (1 - \sin^2 \alpha \cos^2 \varphi_1) / \cos \alpha \). For the planar four-bar the function is given in (17.35), and for the Bennett mechanism it is given in (6.16).

In what follows, only those mechanisms are considered which satisfy the condition \( i(\varphi_1) \neq 0 \). In words: If \( \dot{\varphi}_2 \neq 0 \), then \( \dot{\varphi}_1 \neq 0 \) independent of \( \varphi_1 \). Imagine that in a single-degree-of-freedom mechanism with a known function \( i(\varphi_1) \neq 0 \) a rotor with moment of inertia \( J_1 \) is mounted on the input shaft and that another rotor with moment of inertia \( J_2 \) is mounted on the output shaft. Compared with these rotors the masses and moments of inertia of both shafts and of the coupling mechanism are assumed to be negligible. Furthermore, it is assumed that no external torques are applied to
the shafts and that no springs and dampers are present in the entire system. Under these conditions the mechanism is idling with time-varying angular velocities $\dot{\varphi}_1$ and $\dot{\varphi}_2$. The law of conservation of kinetic energy requires that $J_1\dot{\varphi}_1^2 + J_2\dot{\varphi}_2^2 \equiv 2T = \text{const}$. The constant is given by the initial condition on $\dot{\varphi}_1$. The solution for $\dot{\varphi}_1$ is

$$\dot{\varphi}_1 = \sqrt{\frac{2}{J_1 + J_2} \left( \frac{T}{\omega_0^2} \right)}.$$

(19.1)

The graph of this function in a $\varphi, \dot{\varphi}$-diagram is called phase curve of the motion. Except for the special case $i(\varphi_1) = \text{const}$ which is typical for gear trains the angular velocity $\dot{\varphi}_1$ is not constant.

19.2 The General Problem of Dynamics

Subject of investigation for the rest of this chapter are joint-connected spatial mechanisms without dry friction in the joints. The system structure is arbitrary (serial chain, tree structure, single-loop or multiloop systems). Holonomic joints of arbitrary nature are taken into consideration. Most systems are joint-connected to a frame which is fixed in inertial space. In this chapter the more general case is considered that the system is joint-connected to a carrier body labeled body 0 which is moving relative to inertial space according to prescribed functions of time. Since this motion is prescribed, neither the center of mass nor the mass nor moments of inertia of body 0 are of concern. The body is represented by a moving basis $e_0^0$. The prescribed functions of time for body 0 are the position vector $r_0(t)$, the velocity $\dot{r}_0(t)$ and the acceleration $\ddot{r}_0(t)$ of the origin of basis $e_0^0$ relative to an inertial basis $e$, the direction cosine matrix $A_0(t)$ defined by the equation $e = A_0 e_0^0$, the angular velocity $\omega_0(t)$ and the angular acceleration $\dot{\omega}_0(t)$ relative to basis $e$. Since the motion of $e_0^0$ is prescribed, virtual displacements and virtual changes of velocity are zero. In the special case that body 0 is at rest the matrix $A_0(t)$ is the unit matrix and the other five vectorial quantities are identically zero. This means that $e_0^0$ permanently coincides with $e$.

The general system under consideration is subject to arbitrary external forces and torques as well as to arbitrary internal forces and torques caused by springs, dampers or actuators (active elements) connecting bodies of the system. Let $F \geq 1$ (arbitrary) be the total degree of freedom so that the position of the system in basis $e$ is specified by $F$ independent variables (generalized coordinates). Let $q = [q_1 \ldots q_F]^T$ be the column matrix of suitably chosen variables. At this point it is unnecessary to specify whether