11. CONSTRUCTION OF ELLIPTIC CURVES OF LARGE RANK

11.1. Néron’s specialisation theorem.

Let \( k \) be a number field and \( A \) an abelian variety over \( K = k(T_1, \ldots, T_n) \) where the \( T_i \) are indeterminates over \( k \). By a theorem of Néron ([N1]), \( A(K) \) is a finitely generated group.

**Theorem.** (Néron). There is an abelian variety \( A_t \) over \( k \), of the same dimension as \( A \), such that
\[
\text{rank} A_t(k) \geq \text{rank} A(K).
\]

The variety \( A_t \) is obtained by “specialising” \( A \). This is done as follows: since \( K \) is the function field of \( \mathbb{P}_n \) and \( A \) is defined over \( K \), \( A \) comes from some abelian scheme \( A_U \) over a non-empty open subset \( U \) of \( \mathbb{P}_n \). Let \( s_1, \ldots, s_n \in A(K) \) generate \( A(K) \). One can view the \( s_i \)'s as rational sections of the abelian scheme \( \pi : A_U \to U \). By replacing \( U \) by a smaller open set, one can also assume that the \( s_i \) are morphisms [in fact, such a replacement is not necessary: since \( U \) is smooth, the \( s_i \) can be proved to be everywhere regular on \( U \)]. Hence if \( t \) is any rational point of \( U \) and \( s \in A(K) \), \( s(t) \) is a well defined rational point of \( A_t = \pi^{-1}(t) \). The map \( s \mapsto s(t) \) is a homomorphism \( \phi_t : A(K) \to A_t(k) \).

The precise form of Néron’s theorem is:

**Theorem.** The set of \( t \in U(k) \) for which \( \phi_t \) is not injective is thin.

Therefore outside a thin set, \( \text{rank} A_t(k) \geq \text{rank} A(K) \).

We shall use the following criterion on abelian groups.

**Criterion.** Let \( \phi : M \to N \) be a homomorphism of abelian groups, and \( n \) an integer \( \geq 2 \). We assume that,
1) \( M \) is finitely generated,
2) \( M/nM \to N/nN \) is injective,
3) \( \phi \) is injective on the torsion group of \( M \),
4) \( \phi \) defines an isomorphism of \( M_n = \{ x \in M, nx = 0 \} \) onto \( N_n = \{ x \in N, nx = 0 \} \).

Then \( \phi \) is injective.

**Proof.**

The kernel \( I \) of \( \phi \) is of finite type from (1), and torsion free, from (3). We shall show that \( I = nI \), which implies \( I = 0 \).
Let $x \in I$. We have $\phi(x) = 0 \in nN$, therefore, from (2), $x \in nM$. We write $x = ny$, with $y \in M$. Then $n\phi(y) = \phi(x) = 0$, that is to say $\phi(y) \in N_n$. From (4), there is $z \in M_n$ with $\phi(z) = \phi(y)$. Since $nz = 0$, we have $x = n(y - z)$. But $y - z \in I$, therefore $x \in nI$.

**Proof of Néron's theorem.**

We use the criterion for $\phi_y : A(K) \to A_t(k)$, with an arbitrary choice of $n \geq 2$. Condition (1) is known. Condition (3) follows from a general fact on abelian schemes: if $s$ is a section of order exactly $n$, $s(t)$ is also of order $n$ provided that the residue characteristic at $t$ does not divide $n$ - but here we are in characteristic 0. (The $n$-division points give a sub-scheme $A_n$ étale over $U$. A section of this scheme which is non-zero at the generic point is non-zero elsewhere.)

To check condition (4), we have to prove that there are no more $n$-division points in $A_t$ than in $A$. Take the subscheme $A_n$ which is the kernel of multiplication by $n : A \to A$. As it is étale over the base $U$, we can decompose it into a disjoint sum of irreducible subschemes

$$A_n = \bigsqcup B_\alpha.$$ 

Let $d_i$ be the degree of the projection $B_i \to U$, and let $I$ (resp. $J$) be the set of $i$'s with $d_i = 1$ (resp. $d_i \geq 2$). If $t$ is chosen outside the thin set $\bigcup_{i \in J} \pi(B_i(k))$, we have

$$|A_{t,n}(k)| = |I| = |A_n(K)|,$$

as desired.

To check condition (2), we have to show that, for $t$ outside a suitable thin set, the map

$$A(K)/nA(K) \to A_t(k)/nA_t(k)$$

is injective. It is enough to prove that, for every $\sigma \in A(K)$, $\sigma \notin nA(K)$, there is a thin set $\Omega_\sigma$ such that $\sigma(t) \notin nA_t(k)$ for all $t \notin \Omega_\sigma$. (We construct this for a finite number of $\sigma$: some set of representatives of the classes modulo $n$).

The image of $\sigma : U \to A_U$ is a subvariety of the scheme $A_U$; let $V_{n,\sigma}$ be the inverse image by the morphism $n : A_U \to A_U$. Then $V_{n,\sigma}$ is a covering of $U$, and the hypothesis $\sigma \notin nA(K)$ translates to $V_{n,\sigma}$ having no rational section. We then take $\Omega_\sigma$ to be the image of the rational points of $V_{n,\sigma}(k)$. This concludes the proof.