3

Pathfollowing of Curves of Local Minimizers

3.1 PRELIMINARY OUTLINE

We consider the following one-parametric optimization problem (cf. (1.1.1)):

\[ P(t) \colon \min_{x \in M(t)} \{ f(x, t) \}, \quad t \in [t_A, t_B], \tag{3.1.1} \]

where

\[ M(t) := \{ x \in \mathbb{R}^n | h_i(x, t) = 0, \quad i \in I, \quad g_j(x, t) \leq 0, \quad j \in J \}, \tag{3.1.2} \]

with \( I := \{1, \ldots, m\}, J := \{1, \ldots, s\}, m < n, \) and \( t_A < t_B. \)

In this chapter we assume:

(E1) There exists a continuous function \( x \colon [t_A, t_B] \to \mathbb{R}^n \) such that \( x(t) \) is a local minimizer for \( P(t) \).

(E2) \( x(t_A) \) is known.

(V1) There exists a neighbourhood \( U \) of \( \{(x(t), t) / t \in [t_A, t_B] \} \subset \mathbb{R}^n \times [t_A, t_B] \) such that for all \( (x, t) \in U \) the functions \( f, g_i \) and \( h_j \) \((i = 1, \ldots, m; j = 1, \ldots, s)\) are twice continuously differentiable with respect to \( x \).

(V2) The LICQ is satisfied at \( x(t) \) for each \( t \in [t_A, t_B] \) (cf. Definition 2.3.1).

Assumptions (V1) and (V2) imply that there exist functions \( \lambda \colon [t_A, t_B] \to \mathbb{R}^m \), \( \mu \colon [t_A, t_B] \to \mathbb{R}^s \), which are uniquely defined, such that \( (x(t), \lambda(t), \mu(t)) \) satisfies the KKT conditions (cf. Definition 2.4.2). Additionally, we need the following assumption (the so-called strong second-order sufficient condition):

(V3) \( D_x^2 L(z(t)) | T^{+}_{x(t)}M(t) \) is positive definite for all \( t \in [t_A, t_B] \), where \( z = (x, t) \) (in particular \( z(t) = (x(t), t) \)).
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\[ T^+_{x(t)} M(t) := \bigcap_{i \in I} \text{Ker} \, D_x h_i(z(t)) \cap \bigcap_{j \in J^+(z(t))} \text{Ker} \, d_x g_j(z(t)), \]

\[ J^+(z(t)) := \{ j \in J_\partial(z(t)) | \mu_j(t) > 0 \} \]

and

\[ L(z) = f(z) + \sum_{i \in I} \lambda_i h_i(z) + \sum_{j \in J} \mu_j g_j(z). \]

It is easy to see that the assumptions (E1), (V1), (V2) and (V3) are satisfied if \((f, H, G) \in F**\) and \(z(t) \in \Sigma_{loc}^1 \cup \Sigma_{loc}^2\) for all \(t \in [t_A, t_B]\), i.e. \(x(t)\) is a local minimizer and a point of type 1 or type 2 (cf. Section 2.5).

From Section 2.5 we know that if we restrict ourselves to the class \(F**\), then the assumptions (E1), (V2) and (V3) are not fulfilled for the interval \([0, 1]\) in general. This is the reason why we consider an arbitrary interval \([t_A, t_B]\). With (E2) we have a starting point for the pathfollowing process. Section 3.2 includes an estimation of the radius of convergence of a general locally convergent algorithm (A) of the following structure. We determine a KKT point \(w := (v, t) = (x, \lambda, \mu, t)\) of \(P(t)\) by means of an algorithm of the following kind:

(A) Start with \(v^0\).

Having \(v^i\), let \(v^{i+1}\) be a KKT point of the problem \(P(v^i, t), i = 0, 1, 2, \ldots\), where

\[
P(v, t): \min_{x} \{ \varphi(x, v, t) g_j(\tilde{x}, t) + D_x g_j(\tilde{x}, t)(x - \tilde{x}) \leq 0, j \in J, \]

\[ h_i(\tilde{x}, t) + D_x h_i(\tilde{x}, t)(x - \tilde{x}) = 0, i \in I \}, \]

with

\[ \varphi : \mathbb{R}^n \times \mathbb{R}^{n+m+s} \times [t_A, t_B] \rightarrow \mathbb{R}. \]

If there is more than one such point, choose \(v^{i+1}\) to be closest in norm to \(v^i\).

Choosing special functions \(\varphi\) we obtain the following algorithms:

(1) Robinson’s method ([187])

\[ \varphi(x, \bar{v}, t) := f(x, t) + \sum_{i \in I} \bar{\lambda}_i (h_i(x, t) - h_i(\bar{x}, t) - D_x h_i(\bar{x}, t)(x - \bar{x})) \]

\[ + \sum_{j \in J} \bar{\mu}_j (g_j(x, t) - g_j(\bar{x}, t) - D_x g_j(\bar{x}, t)(x - \bar{x})). \]

(2) Wilson’s method ([187], [229])

\[ \varphi(x, \bar{v}, t) := D_x f(\bar{x}, t)(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T D_x^2 L(\bar{v}, t)(x - \bar{x}). \]