CHAPTER 1

TWO CLASSICAL EXAMPLES: SUBMERSION THEOREM
AND MORSE LEMMA

For a natural understanding of the notions to be introduced in the next chapters, we recall here two elementary results which are basic in many parts of Geometry and Topology.

(1.1) SUBMERSION THEOREM

Let U be an open neighbourhood of the origin $0 \in \mathbb{R}^n$ and $f: U \to \mathbb{R}^p$ a smooth map such that

(i) $f(0)=0$

(ii) $f$ is a submersion at 0 (i.e. rank $df(0)=p \leq n$). Then there is an open neighbourhood $U_1 \subset U$ of the origin $0 \in \mathbb{R}^n$ and a diffeomorphism $g: U_1 \to \mathbb{R}^n$ such that

(a) $g(0)=0$

(b) $g^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_p)$.

Here $df(0)$ denotes the differential of the map $f$ at the point 0, regarded as a linear mapping between the corresponding tangent spaces $\mathbb{R}^n \to \mathbb{R}^p$ and given by the $p \times n$ matrix of partial derivatives $\frac{\partial f_i}{\partial x_j}(0)$, with $i=1, \ldots, p; j=1, \ldots, n$.

Note that we can reformulate the conclusion (b) of this Theorem in a more classical language as follows: There exists a system of coordinates $(\bar{x}_1, \ldots, \bar{x}_n)$ around the origin $0 \in \mathbb{R}^n$ such that the original map $y=f(x)$ can be written in this new coordinate system in the form

$y_1=\bar{x}_1, \ldots, y_p=\bar{x}_p$.

On the other hand, the geometrical meaning is that a submersion (at a point $x_0$) behaves (around $x_0$) like a linear projection.
The proof of Theorem (1.1) is an easy consequence of the Inverse Function Theorem and the reader who needs more details can find them in the first Chapter of Gibson's book [Gi].

To state the second result, we need the following.

(1.2) DEFINITION

Let \( U \subset \mathbb{R}^n \) be an open set, \( x_0 \in U \) a point and \( f: U \to \mathbb{R} \) a smooth function. The point \( x_0 \) is called a nondegenerate singularity of the function \( f \) if

(i) \( df(x_0) = 0 \)

(ii) the Hessian matrix of \( f \) at the point \( x_0 \) is nondegenerate, i.e.

\[
\text{rank} \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right) = i,j=1,\ldots,n=1.
\]

Sometimes, especially in Algebraic Geometry, such a point \( x_0 \) is called a (nondegenerate) quadratic singularity. This is justified by the following fundamental result.

(1.3) MORSE LEMMA

Let \( U \) be an open neighbourhood of the origin \( 0 \in \mathbb{R}^n \) and \( f: U \to \mathbb{R} \) a smooth function such that

(i) \( f(0) = 0 \)

(ii) \( 0 \) is a nondegenerate singularity of the function \( f \).

Then there is an open neighbourhood \( U_1 \subset U \) of the origin \( 0 \in \mathbb{R}^n \) and a diffeomorphism \( g: U_1 \to \mathbb{R}^n \) such that

(a) \( g(0) = 0 \)

(b) \( f \circ g^{-1}(x_1, \ldots, x_n) = -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2 \) for some integer \( p, 0 \leq p \leq n \), called the index of the singular point \( x_0 = 0 \) of \( f \).

In other words, a smooth function \( f \) behaves around a nondegenerate singularity exactly like a nondegenerate quadratic form.

A proof of Morse Lemma will be given in the sequel (see Chapter 6), but some readers may enjoy the short direct proof