CHAPTER 10
CURVE AND SURFACE SINGULARITIES

§1. WEIERSTRASS NORMAL FORM AND BLOWING-UP

In this Chapter we study some special properties of complex isolated hypersurface singularities \((X,0) \subset (\mathbb{C}^n,0)\) of low dimension, namely plane curve singularities \((n=2)\) and surface singularities \((n=3)\).

In this first section we discuss some general facts which hold in any dimension.

Let \((X,0) \subset (\mathbb{C}^n,0)\) be a hypersurface singularity and \(f=0\) be a defining equation for \((X,0)\). Then \(f \in \mathfrak{m}^n \subset \mathbb{C}[x_1,\ldots,x_n]\) and since \(n \geq 2\) this ring is factorial \((2.15.1)\) one can write \(f=f_1^{a_1}\cdots f_p^{a_p}\) for some prime elements \(f_i \in \mathbb{C}[x_1,\ldots,x_n]\) and some integers \(a_i \geq 1\). The analytic space germ \(X\) is reduced if and only if \(a_1 = \ldots = a_p = 1\). Moreover \(X_i : f_i = 0\) are precisely its irreducible components.

(10.1) LEMMA

(i) For \(n=2\), \((X,0)\) is an isolated curve singularity if and only if \(a_1 = \ldots = a_p = 1\).

(ii) For \(n \geq 3\), \((X,0)\) is an isolated hypersurface singularity implies that \(p=1\), \(a_1 = 1\) (i.e. \(X\) is irreducible).

PROOF

(i) If \(a_i > 1\) for some \(i\), all the partial derivatives \(\frac{\partial f}{\partial x_j}\) are divisible by the factor \(f_i\) and hence any point on the irreducible component \(X_i\) is singular.

Conversely, if \(a_1 = \ldots = a_p = 1\) then \(X\) is reduced, and for a reduced curve singularity \(X\) one clearly has \(\dim \text{Sing}(X) < \dim X\). Therefore \(\dim \text{Sing}(X) \leq 0\).

(ii) If \(p \geq 2\) it follows that any point in the intersection \(X_1 \cap X_2\) is singular and hence
(10.2) EXERCISE (WHITNEY UMBRELLA)

Show that the surface $X: x^2 - zy^2 = 0$ in $\mathbb{R}^3$ has the following properties

(i) $(X, 0)$ is not an isolated singularity
(ii) $(X, 0)$ is irreducible.

This shows that the converse of (10.1.ii) does not hold. In fact (10.1.ii) can be strengthened to the next.

(10.3) PROPOSITION

For a normal analytic set germ $(X, 0)$, its singular set $(S_X, 0)$ satisfies $\dim S_X < \dim X - 1$.

The converse implication is true when $(X, 0)$ is a complete intersection singularity.

For the definition of normality and a proof of this result we refer to [KK] p. 315.

Next we present a famous classical result: Weierstrass Preparation Theorem, see for instance [BK], p. 338. The statement (weaker than the usual one) and the proof (quite unusual) of this result given below are in the spirit of the theory developed in our book.

(10.4) PROPOSITION (WEIERSTRASS PREPARATION THEOREM)

Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a finitely determined function germ and let $s \geq 1$ be the order of $f$. Then $f$ is $K$-equivalent to a Weierstrass normal form $f_0$ given by the formula

$$f_0(x_1, \ldots, x_n) = x_1^s + a_2(\overline{x})x_1^{s-2} + \ldots + a_s(\overline{x})$$

where $\overline{x} = (x_2, \ldots, x_n)$ and $a_2, \ldots, a_s$ are polynomials in $E_{n-1}$ such that $\text{ord} (a_i) \geq i$.

Hence $f_0$ is a monic polynomial of degree $s = \text{ord} (f)$ in $x_1$ with coefficients in $E_{n-1}$.

Geometrically, the hypersurface singularity $X: f_0 = 0$ can be regarded as an $s$-sheeted branched covering space over $(\mathbb{C}^{n-1}, 0)$ via the projection.