§1. FURTHER INVARIANTS FOR SINGULARITIES: BOARDMAN SYMBOL AND HILBERT-SAMUEL FUNCTION

In this Chapter we present the $K$-classification of $K$-simple equidimensional map germs $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Such a map germ $f$ is finitely $K$-determined exactly when the fiber $f^{-1}(0)$ is 0-dimensional, and hence a complete intersection. This remark explains the title of this Chapter.

The listing of the normal forms in this situation is due to Giusti [Gt], but the complete list of their specializations was later found by the author and Gibson [DG1], [DG2] (compare to [AGV], p. 172 and [Ln] p. 130).

We start this section by introducing an important numerical invariant or, rather, a sequence of numerical invariants for any map germ $f: (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, namely the Boardman symbol. For any finitely generated ideal $I \subseteq \mathbb{K}^n$, we define the $s$-th Jacobian extension of $I$ to be the ideal $\Xi^s I = I + I'$, where $I'$ is the ideal in $\mathbb{K}^n$ generated by all $(n-s+1) \times (n-s+1)$ minors of the Jacobian matrix

$$
\left( \begin{array}{c}
\frac{\partial f_i}{\partial x_j} \\
\frac{\partial x_j}{\partial x_i}
\end{array} \right)_{i=1,\ldots,p}^{j=1,\ldots,n}
$$

with $f_1, \ldots, f_p$ being a system of generators for the ideal $I$.

Elementary Linear Algebra shows that one has a sequence of inclusions

$$
I = \Delta^0 I \subseteq \Delta^1 I \subseteq \cdots \subseteq \Delta^n I
$$

Suppose now that the ideal $I$ is proper (i.e. $I \not\subseteq \mathbb{K}_n$).

Then the critical Jacobian extension of $I$ is the last ideal $\Delta^i I$ in the sequence (9.1) which is proper. This ideal
Δ^1 I has in turn its critical Jacobian extension Δ^2 Δ^1 I and so on. The sequence of nonnegative integers (i_1, i_2, ...) obtained in this way is called the Boardman symbol of the ideal I.

(9.2) DEFINITION

The Boardman symbol of a map germ \( f \in E^0_{n,p} \) is the Boardman symbol of the ideal \( I_f \) generated by the components \( f_1, \ldots, f_p \) of \( f \) in \( E_n \).

Sometimes the Boardman symbol \( (i_1, i_2, \ldots) \) of the map germ \( f \) is denoted by \( \Sigma f = \Sigma i_1, i_2, \ldots \) and the germ \( f \) is called a map germ of type \( i_1, i_2, \ldots \).

If we take only the first \( k \) integers in the Boardman symbol of a map germ we obtain the \( k \)-th order Boardman symbol \( \Sigma i_1, \ldots, i_k \).

(9.3) EXERCISE

Show that for the simple function singularities in (8.26) one has:

\[ \Sigma(A_k) = (n, 1, \ldots, 1, 0, 0, \ldots) \] with \( (k-1) \)-repeated 1's.

\[ \Sigma(D_k) = (n, 2, 0, 0, \ldots) \]

\[ \Sigma(E_6) = \Sigma(E_7) = (n, 2, 1, 0, 0, \ldots) \]

\[ \Sigma(E_8) = (n, 2, 1, 1, 0, 0, \ldots). \]

Hint: direct computations using the normal forms in \( n \) variables given in (8.26).

Note that all the \( D_k \) singularities have the same Boardman symbol. This indicates that this symbol taken alone is a poor invariant, i.e. it is unable to distinguish among many nonequivalent germs.

On the other hand, consider the next Table (9.4) which contains the Milnor numbers and the third order Boardman symbols of the simple function singularities.