Chapter II Geometric Differential Equations

§ 1. Definition

§ 2. Generalization: modules with connection arising from algebraic geometry

§ 3. Solutions of geometric differential equations; algebraic structure

§ 1. DEFINITION

1.1 - In this chapter we explain in a precise way what should be the differential equations satisfied by G-functions according to the conjecture stated in the introduction: namely, the "geometric" differential equations over $\mathcal{O}$. These are combinations of factors of Picard-Fuchs equations attached to proper smooth varieties defined over $\mathcal{O}(x)$. We study the stability of this class of differential equations under standard operations and show that their solutions in $\mathcal{O}[[x]]$ form a $\mathcal{O}$-vector space stable under Cauchy and Hadamard products (making use of Hodge theory). We shall show in chapter 5 that such solutions are indeed G-functions.

1.2 - Picard-Fuchs differential equations

Let $k$ be a field of characteristic 0. Let $X$ be a smooth $k(x)$-variety. Its algebraic De Rham cohomology groups $H^i_{DR}(X)$ are $k(x)$-vector space endowed with a canonical connection $\nabla : H^i_{DR}(X) \rightarrow H^i_{DR}(X) \otimes \Omega^1_X$, called the Gauss-Manin connection, see for instance [38][39] or 2.1 below. Any non-zero vector in this space provides a differential equation with coefficients in $k[x]$.

For $X$ = a complete curve, everything can be made explicit in the following elementary way (after N. Katz): the only interesting group is $H^1_{DR}(X)$, which can be identified with the group of differentials of the second kind on $X$, modulo the exact ones. Let $t$ be a non-constant function, so that the function field $k(x)(X)$ is a finite extension of $k(x,t)$. Any derivation $D$ of $k(x)$ extends to a derivation $D_t$ on
k(x)(X) by requiring that $D_t(t) = 0$; also the derivation $d/dt$ extends to $k(x)(X)$, and commute with $D_t$. We let $D_t$ act on differentials by $D_t(f dt) = D_t(f) \cdot dt$. The following formulae hold true:

i) $D_t(df) = d(D_t f)$

ii) $\text{res}_p(D_t(f dt)) = D_t(\text{res}_p(f dt))$

iii) $(D_t - D_u)(f dt) = d(fd_u (t))$.

The first one shows that $D_t$ preserves exactness; by ii), $D_t$ acts as a derivation of $H^1_{\text{DR}}(X)$; iii) shows that this action is independent of $t$, and therefore defines the value at $D$ of a connection $\nabla$ on $H^1_{\text{DR}}(X)$: that of Gauss-Manin. Since $H^1_{\text{DR}}(X)$ is of finite dimension $2g$, where $g = \text{genus of } X$, we get for any $\omega \in H^1_{\text{DR}}(X)$ a relation of the form

$\nabla(d/dx)^2g \omega - \sum_{j=0}^{2g-1} \gamma_j \nabla(d/dx)^j \omega = 0$, where $\gamma_j \in k(x)$; multiplying by the common denominator $\delta$ of the $a_i$, we obtain this way an element of $k[x,d/dx]$, namely $\delta(d^{2g}/dx^{2g} - \sum_{j=0}^{2g-1} \gamma_j d^j/dx^j)$.

By way of example, take the hypergeometric differential equation associated to $F(a,b,x)$ and discussed in I.4.4: according to W. Messing (see [40]), these are factors of Picard-Fuchs equations.

1.3 - Geometric differential equations

Let $k[x,d/dx]$ be the Weyl algebra over $k$, sometimes denoted by $A_1(k)$; this is a Noetherian, simple, entire, Euclidean (non-commutative) ring.

DEFINITION. We say that $\Lambda \in A_1(k)$ is a geometric differential equation if $\Lambda$ is a product of factors (in $A_1(k)$) of some Picard-Fuchs differential equations over $k$.

In other words, the game of geometric differential equations consists in picking several Picard-Fuchs equations, decomposing them into irreducible factors and then combining some of these factors in arbitrary order. By way of example, take the polylogarithmic equation discussed in I.4.3. We shall show below that in the definition, it suffices to take Picard-Fuchs