CHAPTER 1

THE LAMINATED MEDIUM

1.1. Introduction

For most solids used in technological applications, the lengths characterizing the ever-present inhomogeneities are much smaller than lengths characterizing the deformation. For the purpose of static and dynamic stress analysis it is then justifiable to describe the mechanical behavior by a classical, homogeneous, isotropic or anisotropic continuum. The corresponding theory, which is termed the effective modulus theory, was discussed in some detail in Part I.

For materials that are fabricated by compounding reinforcing elements and a matrix material to form a directionally reinforced composite material, the characteristic lengths of the medium are substantially larger. Although for most loading conditions the effective modulus theory will still be satisfactory for such composites, it is conceivable that under certain loading conditions, particularly those generating a dynamic response for which the characteristic lengths of the deformation are small, a classical continuum will not give an altogether satisfactory description of the mechanical behavior. For example, the effect of dispersion of time-harmonic waves in an unbounded medium is not described by the effective modulus theory. Dispersion may, however, be of significance in a directionally reinforced composite since it governs the change of shape of a propagating pulse.

For a laminated medium the above considerations have motivated the formulation of a homogeneous continuum model which can describe dynamic effects due to the structuring of the inhomogeneous laminated medium. This Part is devoted to a detailed discussion of a theory for such a homogeneous continuum model. The particular model that is considered here was developed in Refs. 2.1 and 2.2.

The system of governing equations is derived in two stages. The first stage of the derivation involves certain assumptions and operations within the discrete system of layers. In particular, it is assumed that the motions of the individual layers can be described by two-term expansions in a system of local coordinates. The kinematic variables that are introduced in the expansions are defined at the midplanes of the layers only. On the basis of these assumptions, strain
and kinetic energies for representative elements of the layers are computed. In the next stage of the derivation a transition is made from the system of discrete layers to the homogeneous continuum model. The transition is accomplished by defining fields for the kinematical and dynamical variables that are continuous in the coordinate normal to the layering. In discrete planes, which are the midplanes of the layers, the continuous field variables assume the same values as the corresponding field variables that were defined in the discrete system of layers. After averaging, the previously computed kinetic and strain energies can then be interpreted as kinetic and strain energy densities. A subsequent application of Hamilton's principle, where certain continuity relations are included through the use of Lagrangian multipliers, subsequently yields a set of displacement equations of motion. The results of 2.1 demonstrated the advantages of the theory for dynamic problems and justified a study of further ingredients of a complete theory, such as constitutive equations, boundary conditions, uniqueness, etc. These aspects of the theory were discussed in [2.2].

The system of linear field equations, which consists of balance equations, constitutive relations and a constraint condition, resembles the equations of a linear theory of elasticity with microstructure. The resemblance is briefly explored.

1.2. Kinematics and Displacement Equations of Motion

We consider a laminated medium consisting of alternating layers of two homogeneous materials, see Fig. 2.1. It is specified that the field variables and the material parameters in the material whose resistance to deformation is higher (the high-modulus or reinforcing layers) are denoted by subscripts and superscripts \( f \) (fiber). The corresponding quantities in the other layers (the low-modulus or matrix layers) are denoted by subscripts and superscripts \( m \) (matrix). We focus attention on an element of the k-th pair of reinforcing and matrix layers whose midplane positions are defined by \( x_2^{(f,k)} \) and \( x_2^{(m,k)} \), respectively, see Fig. 2.2, and we define two local Cartesian coordinate systems

\[
\left( x_1, x_2^{(f)}, x_3 \right) \quad \text{and} \quad \left( x_1, x_2^{(m)}, x_3 \right).
\]

If the characteristic length of the deformation is larger than the thickness of the layers, then, analogously to plate theories, the displacements in the