1 The Lorentz Transformation

Traditionally, two postulates are put at the beginning of Special Relativity, from which all other results can be derived:

A. The Principle of Relativity
B. The constancy of the speed of light

From these principles the Lorentz transformation may be derived in numerous ways, some more and some less elementary, as is done in most presentations of Relativity. Already from 1910 on, authors occasionally pointed out¹ that the principle of relativity alone already determines almost all of the structure, and in particular implies the existence of a (numerically unspecified) invariant speed. This approach does not concentrate on a single Lorentz transformation but works with the totality of all transformations admitted by the principle of relativity. Thus, group theoretical ideas, on which we are going to elaborate in this book, come in implicitly or explicitly right from the beginning. We therefore here set out to derive the Lorentz transformation in a manner that takes into account this central role of principle A, and take B only to decide between the numbers \(-1, 0, \text{ and } 1\).

To understand the principle of relativity, we have to analyze the concept of ‘inertial systems of reference’, which we do first.

1.1 Inertial Systems

Consider a number of labs in free flight (Fig. 1.1)—we assume we can neglect their mutual interactions (by gravitation, say). Within each of them, Newton’s First Axiom

![Fig. 1.1. Labs in free flight](image)

(the law of inertia) holds, every body with no forces acting on it remains—as judged from the lab—in a state of rest or of uniform rectilinear motion. Such a lab defines an inertial system \(I\). Each (pointlike) event may be recorded by noting its coordinates

\( x, y, z \) with respect to a rectangular Cartesian coordinate system anchored in \( I \) together with the reading \( t \) of a clock attached to \( I \). We shall term this setup an \textit{inertial reference frame}, and we restrict to positively oriented coordinate axes at the moment. It is useful to consider \( t, x, y, z \) as four coordinates \( x^i = (x^0, x^1, x^2, x^3) := (t, x, y, z) \). Time thus appears—at first in a purely formal manner—as a fourth (‘zeroth’) coordinate.

To describe the motion of some point mass with respect to such an inertial system \( I \) it is also helpful to use \textit{space-time diagrams}. (For actual drawings we must restrict to less than three space dimensions, however (see Fig. 1.2).) The consecutive positions of the moving point mass in this diagram make up its \textit{world line}. As one easily convinces oneself, for rectilinear uniform motion the world line is straight, and conversely.

![Space-time diagrams for the motion of a mass point](image)

Our next task is to find the relation between different inertial frames. If \( I \) is inertial, then from experience we know that a reference frame \( \bar{I} \) is again inertial if

\begin{itemize}
  \item[a.] parallely displaced by \( \vec{a} \)
  \item[b.] rotated by \( \vec{\alpha} \)
  \item[c.] moving at constant velocity \( \vec{v} \)
  \item[d.] time delayed by \( a^0 \)
\end{itemize}

Here \( \vec{\alpha} \) is the rotation vector (see later; it may be replaced by any other triple of numbers capable of fixing a rotation), and \( a^0 \) is the time lag between the clocks attached to the two systems; parallel displacement and rotation refer to Euclidean Geometry, valid by experience in every inertial system. One does not, however, obtain new inertial systems by considering systems accelerated against \( I \). We exclude transformations of units of length and time by assuming—as justified from experience—the existence of measuring rods and clocks insensitive to accelerations, which may be used to gauge all inertial frames (cf. sect. 2.8).