3 Lorentz Group, Poincaré Group, and Minkowski Geometry

As a consequence of the Principle of Relativity, the set $\mathcal{P}$ of transformations between inertial systems has a certain mathematical structure: composing two transformations from $\mathcal{P}$ gives a transformation from $\mathcal{P}$ again, and for each transformation from $\mathcal{P}$ there is a unique inverse in $\mathcal{P}$. The set $\mathcal{P}$ therefore forms a group, where the group multiplication law is given by the composition of transformations.

Generally, by a group $\mathcal{G}$ one means a set of elements, $\{g, h, \ldots\}$, where to each ordered pair $(g, h)$ of elements a 'product' $gh$ in $\mathcal{P}$ is assigned such that the following rules (group axioms) hold:

1. $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ (associativity)

2. There exists an element $e \in \mathcal{G}$ such that $eg = ge = g$ for all $g \in \mathcal{G}$ (unit element)

3. For each $g \in \mathcal{G}$ there is an element $g^{-1} \in \mathcal{G}$ such that $g^{-1} g = g g^{-1} = e$. (Inverse)

In our case $\mathcal{G} = \mathcal{P}$, $e$ is the identical transformation and $g^{-1}$ is the inverse transformation. Two things are to be observed:

- A group is given abstractly by its 'multiplication table' which registers the product $gh$ for each pair $g, h$ of elements. The group is called Abelian or commutative if throughout $\mathcal{G}$ one has $gh = hg$. The group $\mathcal{P}$ is not commutative, and its elements are 'numbered' or 'indexed' by 10 parameters that can vary continuously—cf. sect. 1.1.

- The group $\mathcal{P}$ is not given abstractly but as a group of transformations acting on the set $\mathcal{I}$ of inertial frames or on the set $\mathbb{R}^4$ of event coordinates. We shall see that the same abstract group acts (or is realized) in various different ways as a group of transformations on sets of elements (physical objects) of various kinds (inertial frames, event coordinates, events, four-vectors, tensors, spinors, fields, state vectors in Hilbert spaces, ...), so that it will soon become evident that the abstract point of view is very useful.

Although we shall verify the group property of $\mathcal{P}$ explicitly in the exercises to sect. 3.1, let us sketch here an argument why it must be a group on the basis of the Principle of Relativity. (A reader unable to appreciate this kind of 'abstract nonsense' argument should not be discouraged at this point!) Write again $\mathcal{I}$ for the set of all inertial frames and write $\mathcal{E}$ for the set of all space-time events. Then every $I \in \mathcal{I}$ gives, by definition of a frame of reference, a bijective map between $\mathbb{R}^4$ (the set of event coordinates) and $\mathcal{E}$ which we denote by the same letter; thus $I : \mathbb{R}^4 \rightarrow \mathcal{E}$, $\bar{I} : \mathbb{R}^4 \rightarrow \mathcal{E}$.

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etc. Associated to any pair \( I_i, I_j \) of frames is a transition map \( f_{ij} = I_i^{-1} \circ I_j : \mathbb{R}^4 \to \mathbb{R}^4 \). (These are the transformations written so far, beginning with eq. (1.1.1).) They obviously satisfy
\[ f_{ij} \circ f_{jk} = f_{ik}, \quad f_{ij}^{-1} = f_{ji}, \quad f_{ii} = \text{id}. \]

Let \( P(I) \) be the set of all transition maps \( I^{-1} \circ J \) connecting \( I \) to all other frames \( J \). Then the Principle of Relativity implies that this set is the same for all \( I \), i.e., \( P(I) = P(I) = \ldots =: P \). It is easy to deduce from this and the relations for the \( f_{ij} \) just written that \( P \) is a group (of bijections \( \mathbb{R}^4 \to \mathbb{R}^4 \)) under composition of maps as the multiplication. Namely, to show that the composition \( f_{ij} \circ f_{mn} \) also belongs to \( P \) although the adjacent indices do not agree as in the relation above, conclude from \( P(I_m) = P(I_j) \) that there must exist a system \( I_k \) such that \( f_{mn} = f_{mk} \), which makes the relation above applicable.

The group \( P \) acts on event coordinates (i.e., on \( \mathbb{R}^4 \)) but can also be thought of as acting on inertial frames (i.e., on \( I \)) 'from the right' as \( I \mapsto f \circ I \) for \( f \in P \). Note that after singling out any inertial frame \( I_0 \in I \) we have a bijective correspondence between \( I \) and \( P \) by assigning to every \( I \) the unique transition map by which it is obtained from \( I_0 \); but only \( P \) is a group (one cannotmeaningfully multiply inertial systems)!

We therefore have an action of the group on the product space \( I \times \mathbb{R}^4 \), and calling the pairs \((I, (x^i))\) and \((I, (x^i'))\) equivalent iff \( I = I \circ f^{-1} \), \( x^i = f(x^i) \) for some \( f \in P \) allows to identify \( E \) with the quotient \((I \times \mathbb{R}^4)/P\) by this equivalence relation. This construction will allow to transfer properties of \( \mathbb{R}^4 \) relative to the group \( P \) to the event space \( E \) (differentiable structure, affine structure, pseudometric, \ldots). We will then also consider active versions of the transformations, i.e., transformations of \( E \) described by \( I \circ f \circ I^{-1} \), where \( f \in P \); they can also be characterized as leaving invariant the structures just mentioned.

The basic idea behind using the abstract group is that there are systematic mathematical methods for constructing and classifying other realizations once the abstract group structure has been found from one realization as a transformation group. The new objects on which the new realizations act can be used as building blocks in attempts to construct new physical theories such that the Principle of Relativity will automatically hold in them.

In this book our aim is to go on with such a program step by step, becoming acquainted with some of the pertinent methods and kinds of arguments, without however putting too much stress on rigor or completeness.

### 3.1 Lorentz Group and Poincaré Group

In sect. 1.5 we characterized the general Poincaré transformations as being those coordinate transformations
\[ x^i = f^i(x^k) \quad (3.1.1) \]
leaving invariant the line element (1.5.1),
\[ ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = \eta_{hk} \, dx^h dx^k. \quad (3.1.2) \]
Here we have introduced the component matrix of the so-called metric tensor\(^1\),
\[ \eta = (\eta_{hk}) := \text{diag} (1, -1, -1) = (\eta_{ik}), \quad (3.1.3) \]
\(^1\)This name will be explained later.