Applications of a Fruitful Relation between a Dupin Cyclide and a Right Circular Cone

G. Albrecht, München

Abstract

A powerful way of handling a Dupin cyclide is presented. It is based on the concept of inversion, which yields a fruitful relation between the symmetric Dupin horn cyclide, from which all other Dupin cyclides may be obtained by offsetting, and a right circular cone. This relation has two important applications. First, it is used for constructing rational rectangular and triangular Bézier patches on the cyclide. Second, it allows to establish an approximative isometry between cyclide and cone patches, a useful result e.g. for scattered data interpolation techniques on Dupin cyclides.

Key words: Dupin cyclide, cone, inversion, rational Bézier rectangles and triangles, scattered data interpolation.

1. Introduction

Dupin cyclides have proven to be a useful class of surfaces for CAGD. They offer a simple means for augmenting the flexibility of solid modellers since they encompass planes, tori, and the so-called natural quadrics (spheres, right circular cylinders/cones) as special cases (see e.g. [20]), i.e., the surfaces traditionally used in solid modelling systems.

In order to be able to fully utilize Dupin cyclides in this context, intersection algorithms—among other things—must be provided. This problem has been addressed by several authors, see e.g. [11, 16], where Johnstone [11] makes use of the concept of inversion. In engineering applications Dupin cyclides have been found to be particularly important as blending surfaces for planes, cones, cylinders, and spheres. References [3, 9, 10, 16–18] are some contributions concerning this area. Dupin cyclides are of practical use also for 3D motion planning [7] and for creating and manipulating geometrically more complex objects [8].

For using Dupin cyclides in standard CAD systems based on free form surfaces, Bézier or B-spline representations are required for them. Pratt [17] and Kaps [12] present biquadratic rational Bézier representations of a so-called principal cyclide patch, a rectangular surface patch on a Dupin cyclide bounded by lines of curvature, which was first introduced by Martin [15]. Pratt then uses his previous results in the application of Dupin cyclides as blending surfaces [18], i.e.,
he determines the position of the Bézier control points for certain cyclide blends. *Normalized* principal cyclide patches, i.e., rational Bézier patches on a Dupin cyclide having weights equal to 1 at three of four corner points, are studied by Ueda [22], and a NURBS representation of a principal cyclide patch is given by Zhou and Strasser [24].

Also, so-called generalized cyclides [5] or double Blutel surfaces [4], which are projective images of Dupin cyclides, as well as an even more general class of surfaces [6] having—among other things—plane rational curves as parameter lines and containing the generalized cyclides as a special case have found to be promising surfaces for CAGD purposes (see also [19]).

Thus, due to the importance of Dupin cyclides it is desirable to have a very easy access to these surfaces of fourth order. This paper presents a powerful way of handling a symmetric Dupin horn cyclide, from which every other Dupin cyclide may be obtained by offsetting, namely by reducing it to the much simpler surface of a right circular cone. The main idea is to map the symmetric Dupin horn cyclide to a right circular cone through an inversion with respect to one of the real conical points of the cyclide. This yields two different applications. First, based on the mathematical details provided in [1] rational Bézier representations of rectangular and triangular patches on the symmetric Dupin horn cyclide are obtained. Second, a method is presented for relating a cyclide patch and a cone patch such that they are approximately, i.e. up to a prescribed tolerance, isometric. This can e.g. be used for reducing distance measurement on the cyclide to the much simpler problem of measuring distances on the cone, which is a developable surface.

The paper is organized as follows: in Section 2 the principle of inversion and the symmetric Dupin horn cyclide are presented, and the inversion map is applied to the implicit equation of the cyclide. In Section 3.1 rational rectangular biquadratic and rational triangular quartic patches on the symmetric Dupin horn cyclide are constructed. Section 3.2 presents a method for mapping a cyclide patch to a cone patch, such that they approximately have the same inner geometry, i.e. are approximately isometric. Section 4 contains some concluding remarks.

### 2. Principle of Inversion and Dupin Cyclide

In analytical geometry the inversion map (see e.g. [14] pp. 177–181, [21], p. 201) is defined to be the reflection with respect to a circle $S^1$ (in $\mathbb{R}^2$) or a sphere $S^2$ (in $\mathbb{R}^3$) with center $z$ and radius $\rho$, such that a point $x$ and its image $x'$ are related by

$$\overline{zx} \cdot \overline{zx'} = \rho^2. \quad (1)$$

$\overline{zx}$ denotes the Euclidean distance between the points $z$ and $x$, and $S^1$ resp. $S^2$ is called the *circle* resp. *sphere of inversion*, $z$ the *center of inversion* and $\rho$ the *radius of inversion*. For an illustration, see Fig. 1.