XII. Signals and Signal Processing

It is very important that methods be developed to extract signals from the noisy backgrounds that usually accompany them. Signal processing has, therefore, become an important field in communications and in acoustics. The human ear processes the information it receives and does a reasonably good job of filtering. A knowledge of the signal processing techniques that have been developed in the past will also be helpful in understanding the functioning of the human ear.

12.1. Beats and Signals

If two sinusoidal vibrations of similar frequency are added, beats are generated. For instance,

\[ B \left( \cos \omega_1 t + \cos \omega_2 t \right) = 2 B \cos \frac{\omega_2 - \omega_1}{2} t \cos \frac{\omega_2 + \omega_1}{2} t = A(t) \cos \omega_m t, \]  

(1)

where

\[ A(t) = 2 B \cos \frac{\omega_m}{2} t, \]  

(2)

can be interpreted as the slowly varying amplitude of an oscillation of a frequency equal to the mean of the two frequencies \( \omega_2 \) and \( \omega_1 \):

\[ \omega_m = \frac{\omega_2 + \omega_1}{2}. \]  

(3)

The frequency of the amplitude variation is

\[ \frac{\Delta \omega}{2} = \frac{\omega_2 - \omega_1}{2}. \]  

(4)

The resulting phenomenon is called a beat (see Fig. 1). The number of beats per second is equal to the number of positive and negative maxima or the number of zeroes per second of the amplitude function \( A(t) \), which is equal to the difference frequency \( \frac{\Delta \omega}{2 \pi} \) (and not half of it). Because of the carrier, a change in sign of \( A(t) \) from positive to negative displaces the maximum from the positive half wave of the carrier to its negative half wave and, thus, has practically no effect on the envelope of the beat. Note that if the two vibrations have the same amplitude, and if \( f(t) \) is the instantaneous amplitude of the resulting oscillation, \( f(-t) = f(t) \) and not \( -f(t) \) (see Fig. 1b) where \( t = 0 \) is the time for the zero value of the envelope.
Fourier analysis of a beat should lead to the resolution of the two sinusoidal vibrations of frequency $\omega_1$ and $\omega_2$. That the human ear perceives beats is a consequence of its limited integration. The ear does not perform a true Fourier analysis but, rather, performs a weighted analysis and integrates only over time intervals of about $1/20$ second. Consequently, it perceives the beat as a swelling on and off of the loudness.

If $n$ oscillations of the same amplitude are added whose frequencies deviate successively by $\Delta \omega$, we obtain a sequence of major and minor maxima. The computation is as follows:

$$s(t) = \sum_{\nu=0}^{n-1} \sin(\omega_1 + \nu \Delta \omega)t = \text{Im}(e^{j\omega_1 t}(1 + e^{j\Delta \omega t} + e^{2j\Delta \omega t} + \ldots + e^{(n-1)j\Delta \omega t}))$$

$$= \text{Im}\left(e^{j\omega_1 t} \frac{1-e^{jn\Delta \omega t}}{1-e^{j\Delta \omega t}}\right) = \text{Im}\left(e^{j\omega_1 t} \frac{e^{-j\frac{n-1}{2}\Delta \omega t}}{e^{-j\Delta \omega t/2}} \frac{e^{j\frac{n\Delta \omega t}{2}} - e^{jn\Delta \omega t/2}}{e^{-j\Delta \omega t/2} - e^{j\Delta \omega t/2}}\right)$$

$$= \frac{\sin(n \Delta \omega t/2)}{n \sin(\Delta \omega t/2)} n \sin\left[(\omega_1 + \frac{n-1}{2} \Delta \omega) t\right] = \frac{n \sin n \Delta \omega t/2}{n \sin \Delta \omega t/2} \sin \omega_m t,$$

where

$$\omega_m = \omega_1 + \frac{n-1}{2} \Delta \omega = \omega_1 + \frac{\omega_2 - \omega_1}{2} = \frac{\omega_1 + \omega_2}{2}$$

is the mean frequency.

At the time $t = 0$, all the oscillations are in phase and add up to a major maximum. But, because of their different frequencies, their phases run...