VI. The Laplace Transform

6.1. One-Sided Time Functions and Enforced Convergence

The Fourier transform exists only if the function that is to be transformed is absolutely integrable. This means that \( s(t) \) or \( S(\omega) \) must decrease faster than

\[
\begin{align*}
  s(t) &\to 1/t \\
  S(\omega) &\to 1/\omega
\end{align*}
\]

as \( t \) or \( \omega \) approach infinity. If methods of analysis that are based on Fourier transforms are to lead to practical results, the initial values of all variables \( t = -\infty \) must be zero; and they must be zero, too, in the final state of the system. Thus, the standard Fourier analysis is not capable of any initial values or any final values of the functions other than zero. There are two possibilities for dealing with this situation. It can be assumed that all time functions are zero initially and increase to the given values in a very short time interval and finally decrease abruptly to zero when the time exceeds a certain value. Thus, the time functions are truncated (made zero), for \( t < -T \) and for \( t > +T \). This procedure will be investigated in Chapter VII. The second procedure is to assume that the function \( s(t) \) is generated at \( t = 0 \):

\[
s(t) = 0, \quad t < 0,
\]

and to introduce an infinitely small amount of damping

\[
s(t) \to \lim_{\delta \to 0} s(t) e^{-\delta t}, \quad t > 0
\]

as we have already done in considering the step function and the switched on sinusoidal vibration. The Fourier transform then exists,

\[
\tilde{S}(\omega) = \int_{0}^{\infty} s(t) e^{j \omega t} dt = \int_{0}^{\infty} s(t) e^{-p t} dt
\]

or

\[
S(p) = \int_{0}^{\infty} s(t) e^{-p t} dt,
\]

where

\[
p = j \omega + \delta = j (\omega - j \delta) = j \bar{\omega}.
\]

Because the convergence factor \( e^{-\delta t} \) becomes a divergence factor for negative values of \( t \), \( s(t) \) must be assumed to be zero for negative values of \( t \).
The inverse transformation, then, is
\[ s(t) e^{-\delta t} = \int_{-\infty}^{\infty} \mathcal{S}(\omega) e^{j\omega t} \frac{d\omega}{2\pi}, \tag{8} \]
and
\[ s(t) = \int_{-\infty}^{\infty} \mathcal{S}(\omega) e^{\delta t + j\omega t} \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \mathcal{S}(\omega) e^{j\omega t} \frac{d(\omega - j\delta)}{2\pi} \tag{9} \]
because \( \delta \) is constant. If \( p = j\omega \) is introduced as integration variable,
\[ s(t) = \int_{-j\infty}^{j\infty} \mathcal{S}(p) e^{pt} \frac{dp}{2\pi j}, \tag{10} \]
where
\[ \mathcal{S}(p) = \int_{0}^{\infty} s(t) e^{-pt} dt. \tag{11} \]
In the last line, \( \mathcal{S}(p) \) has been written for \( \mathcal{S}(\omega - j\delta) \) to simplify writing\(^1\).

Note once more that the Fourier integral limits the same time function (interference pattern) to constant amplitude sinusoids, whereas the Laplace transform (because of the possibility of deforming the path of integration in the complex plane) builds the same patterns from endless mixtures of growing or decaying sinusoids. The two last expressions represent the Laplace transformation and its inverse transformation. The damping (\( \delta \)) has to be large enough to enforce convergence. This means that, for passive systems or solutions for passive systems, \( \delta \) must be greater than zero. The transformation is a unilateral one, the integration range for \( t \) extends only from \( 0 \) to \( \infty \).

The original time function is obtained regardless of how much damping has been assumed in the convergence factor \( e^{-\delta t} \). From a mathematical point of view, it is convenient to assume \( \delta \to 0 \) so that the path of integration runs along the right-hand side of, but infinitely close to, the imaginary axis.

### 6.2. Computation Rules

The first step in modifying Fourier analysis for practical application—the introduction of the damping factor \( e^{-\delta t} \)—connects Fourier analysis with reality by expressing the fact that, as time passes by, all vibrations and oscillations die out. The second step consists of the evaluation of the integral. For positive values of the time, \( t > 0 \), the path of integration\(^2\) may be closed

\(^1\) It has already been pointed out in the introduction (symbols) that the operator or variable \( p \) of the Laplace transform will be without a bar. It is a logical consequence that also functions of \( p \) like \( \mathcal{S}(p) \) will be written without bars, as if they were real functions of a real variable \( p \).

\(^2\) Jordan’s theorem (section 3.9) states that if \( f(z) \) is regular for \( \text{Im}(z) \geq 0 \) except at a limited number of poles, and if \( f(z) \to 0 \) on a semi-circle \( T \) in the upper half-plane as \( R \to \infty \) for all values of \( \text{arg} \, z = \varphi \) such that \( 0 \leq \varphi \leq \pi \) and if \( m > 0 \),
\[ \int_{T} f(z) e^{m \varphi z} dz \to 0, \text{ as } R \to \infty. \]