III. Linearization of Nonlinear Programming Problems

A. Linear Approximations and Linear Programming

In the absence of general algorithms for nonlinear programming problems, it lies near at hand to explore the possibilities of approximate solution by linearization. If the nonlinear functions of an MP problem can be replaced by piecewise linear functions, these approximations may be expressed in such a way that the whole problem is turned into a case of linear programming. Applying the simplex method we get a solution which—assuming that all functions involved have the concavity or convexity properties required—is an approximation of the true global maximum or minimum.

For example, let the side conditions of a problem be linear whereas the preference function \( f = f (x_1, x_2, \ldots, x_n) \) is a nonlinear differentiable function. A continuous function of a single variable can always be approximated by a piecewise linear function. If \( f \) is separable, i.e., if it can be written in the form

\[
f = f_1 (x_1) + f_2 (x_2) + \ldots + f_n (x_n),
\]

this procedure can be applied to each of the terms \( f_j (x_j) \) as shown in Fig. 3. The closer the points of interpolation are placed, the better the approximation.

The next step is to reformulate the problem as an LP problem, making use of the fact that \( f \) is now linear within each of the successive intervals of the \( x_j \). This can be done in several ways. In the following
we shall deal with two such procedures, the method of partitioning the variables and the method known as separable programming\(^1\). Either method is equally applicable to problems where the piecewise linear function is not an approximation but represents the true preference function; in such a case the simplex solution to the linearized problem is the true solution, not an approximation.

### B. Partitioning of Variables

1. Consider the following problem of optimal utilization of machine capacities: A company manufactures two products which have to be processed on the same two machines whose capacities set limits to production. \(x_1\) and \(x_2\) units are produced per period and sold at prices which depend on the quantities to be sold, \(p_1 = 12 - x_1\) and \(p_2 = 13 - 2x_2\) ($ per unit); the variable costs of labour, raw materials, and other current inputs are 4 and 3 respectively per unit of product. Hence total gross profit is

\[
 f = (12 - x_1) x_1 + (13 - 2x_2) x_2 - 4x_1 - 3x_2
\]

which is to be maximized subject to the capacity restrictions, one for each machine:

\[
 \begin{align*}
 f &= 8x_1 - x_1^2 + 10x_2 - 2x_2^2 = \max \\
 x_1 + x_2 &\leq 5 \\
 x_1 + 2x_2 &\leq 8 \\
 x_1, x_2 &\geq 0
\end{align*}
\]

(1)

where the coefficients of \(x_1\) and \(x_2\) in the capacity restrictions represent machine hours required per unit of product and the right-hand sides are machine hours available per period.

Since the linear functions in the side conditions, \(g_1 = 5 - x_1 - x_2\) and \(g_2 = 8 - x_1 - 2x_2\), are concave and each term in the function to be maximized is also concave, the Kuhn-Tucker conditions are necessary and sufficient for a global maximum of \(f\) subject to the inequality constraints.

To find an approximate solution, we first separate the preference function into \(f_1 (x_1) = 8x_1 - x_1^2\) and \(f_2 (x_2) = 10x_2 - 2x_2^2\), and next approximate each of these by piecewise linear functions. Starting with \(f_1 (x_1)\), it is clear from the capacity restrictions that the nonnegative variable \(x_1\) cannot be greater than 5, so we must interpolate between

\(^1\) See e.g. E. M. L. Beale (1968), Ch. 14; G. B. Dantzig (1963), Ch. 24; S. Vajda (1961), Ch. 12; and R. Henn and H. P. Künzi (1968), Vol. II, Ch. 6.