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The Analyticity Program in Axiomatic Quantum Field Theory

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Summary. This paper reviews some topics and results of the linear and the nonlinear programs of Axiomatic Quantum Field Theory. Applications to Constructive Field Theory are also discussed.

11.1 Introduction

This review treats analyticity in energy-momentum variables (dual to time and space variables respectively) in traditional Quantum Field Theory (QFT). It roughly covers the period 1960–1990. In 1960, analyticity is clearly recognized as an important property of collision amplitudes in relativistic quantum physics, a fact which leads to a number of rigorous works and results, in various approaches, in successive steps in the 1960s through 1980s. A more recent result related to this analyticity program concerns analyticity in angular momentum (presented in this book by G.A. Viano).

Viewed from a current perspective, the analysis includes a number of basic results on the general structure of 2-body and multiparticle collision amplitudes. It has also been related to, and at the origin of some, deep mathematical problems and results in the domain of analytic functions of several complex variables and of what is called today "microlocal analysis". On the negative side, some of the results so far apply only to "massive" theories (particles with strictly positive masses only), whose existence in space-time dimension 4 is very doubtful. Nevertheless, it is hoped that the analysis will keep some value in more general theories.

The paper will mainly treat the *analyticity program in the axiomatic QFT approach*, in which Jacques Bros has played a major role. The related approach of Constructive QFT, in which his ideas also played a direct role, will be briefly presented at the end. Other approaches (analysis of Feynman integrals in perturbative QFT, "axiomatic" S-matrix theory) will be briefly mentioned in the course of the talk. Concerning the axiomatic analyticity program, two types of methods and results can be distinguished; they lead respectively to what we shall call *microlocal* and *global analyticity properties* of the N-point functions of the fields and $m \to n$ particle collision amplitudes. As a general rule, our survey will provide more information

about the former than about the latter; in particular, such important and well-known global results of the case $N = 4$ as the dispersion relations for two-particle collision amplitudes and the rigorous derivation of the Froissart bounds (and other types of bounds) by A. Martin will only be briefly mentioned at the relevant place of our survey.

For conciseness, references to original works are omitted. For details and references, see the book by the author "Scattering in Quantum Field Theories: the Axiomatic and Constructive Approaches", Princeton University Press, 1991. Among contributions by many authors, most crucial works on the topics treated are due to A. S. Wightman et al in Sect. 1, J. Bros in Sect. 2, J. Bros, H. Epstein, V. Glaser in Sect. 3, H.P. Stapp in Sect. 4, J. Bros in Sect. 5 and J. Magnen in Sect. 6.

11.2 The axiomatic framework

11.2.1 Wightman axioms and Asymptotic Completeness

We denote by $x = (x_0, \vec{x})$ a (real) point in Minkowski space-time with respective time and space components x_0 and $\vec{x}$ (in a given Lorentz frame), the Minkowskian squared pseudo-norm of x being $x^2 = x_0^2 - \vec{x}^2$. Besides the usual physical space-time dimension $d = 4$, possible values 2 or 3 will also be considered. In all that follows, the unit system is such that the velocity c of light is equal to 1. Energy-momentum variables, dual (by Fourier transformation) to time and space variables respectively, are denoted by $p = (p_0, \vec{p})$; the corresponding squared mass variable is $p^2 = p_0^2 - \vec{p}^2$.

We describe below the Wightman axiomatic framework, though alternative frameworks such as the one of "Local Quantum Physics" based on the Araki-Haag-Kastler axioms, may be used similarly for present purposes with only minor changes. For simplicity, we consider a theory with only one basic (neutral, scalar) field A, defined on space-time as an operator-valued distribution: for each test function f, $A(f)$ (i.e. formally the integral $\int A(x)f(x)dx$) is an operator in a Hilbert space $\mathcal{H}$ of states (by abuse, we will also speak of $A(x)$, for a given point x, as an "operator in $\mathcal{H}$"). A physical state is more precisely represented by a (normalized) vector in $\mathcal{H}$ modulo scalar multiples. It has to be physically understood as "sub specie aeternitatis" (i.e. "with all its evolution", the Heisenberg picture of quantum mechanics being always adopted).

It is assumed that there exists in $\mathcal{H}$ a representation of the Poincaré group (semi-direct product of pure Lorentz transformations and space-time translations).

The Wightman axioms

– local commutativity : the operators $A(x)$ and $A(y)$ commute if $x - y$ is space-like (i.e. $(x-y)^2 < 0$).

– spectral condition (= positivity of the energy in relativistic form): the spectrum of the energy-momentum operators (infinitesimal generators of space-time translations) is contained in the future cone V_+ ($p^2 \geq 0, p_0 \geq 0$).

In a theory "with mass gap", the spectrum is more precisely assumed to be contained in the union of the origin (that will correspond to the vacuum vector introduced next), of one or more discrete mass-shell hyperboloids $H_+(m_i)$ ($p^2 = m_i^2$,