Biorthogonal Wavelets Associated with Two-Dimensional Interpolatory Function

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Abstract. To construct biorthogonal wavelets from two-dimensional interpolatory function, a large amount of computation is involved in traditional method. In this paper, a method is developed for constructing the biorthogonal wavelets. Masks of the biorthogonal wavelets are given explicitly. Neither the Gram-Schmidt processing nor the inverse of a nonsingular polynomial matrix is needed.

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1. Introduction

During the past few years, the construction of interpolatory scaling function has become of increasing interest (see, e.g. [1, 2, 3, 4]). However, there still does not exist a simple method to construct biorthogonal wavelets from the associated two-dimensional interpolatory function. In this paper, a method is presented for constructing biorthogonal wavelets from two-dimensional interpolatory function. Biorthogonal wavelet masks can be constructed explicitly.

Let \( \phi(x) \in L^2(R^2) \), which satisfies the following refinable equation

\[
\phi(x) = \sum_{\alpha \in Z^2} p_\alpha \phi(2x - \alpha),
\]

and let

\[
V_0 = \text{Span}_{L^2(R^2)} \{ \phi(x - \alpha), \; \alpha \in Z^2 \}, \; V_k = \{ f(2^k x - \alpha), \; f \in V_0, \; \alpha \in Z^2 \}.
\]

If \( \hat{\phi}(0) \neq 0 \), it was shown in [5] (see also [6]) that \( \{ V_k \}_{k \in Z} \), the sequence of subspace of \( L^2(R^2) \), satisfies

\[
\bigcup_{k \in Z} V_k = L^2(R^2), \quad \bigcap_{k \in Z} V_k = \{ 0 \}.
\]
If $\phi(x)$ and its shifts form a Riesz basis of $V_0$, the sequence of the subspaces \{\text{\textit{V}}_k\}_{k\in \mathbb{Z}}$ forms a multiresolution analysis (MRA) of $L^2(\mathbb{R}^2)$. $\phi(x)$ is called a scaling function.

If $\phi(x)$ is a continuous two-dimensional scaling function, which satisfies

$$\phi(\alpha) = \delta_{0,\alpha}, \quad \alpha \in \mathbb{Z}^2,$$

we say that $\phi(x)$ is an interpolatory scaling function.

In practical image processing, images are often first represented by sampling space $V_k$. When the pixel values of an image $f$ are given, an image is normally (or easily) represented by

$$f_k = 2^k \sum_{\alpha \in \mathbb{Z}^2} f(\frac{\alpha}{2^k})\phi(2^k x - \alpha)$$

for a certain dilation level $k$. However, to apply the decomposition and reconstruction algorithm, one should use the function [3]

$$\sum_{\alpha \in \mathbb{Z}^2} < f_k, 2^k \phi(2^k x - \alpha) > 2^k \phi(2^k - \alpha).$$

This function is not the function $f_k$, unless the refinable function $\phi$ satisfies the condition (1.2). Hence, by using the sampling space generated by interpolatory refinable function, one simplifies (or reduces the errors of) the first step of the decomposition and reconstruction algorithm (see [3]). During the past few years, some excellent results on the construction of interpolatory functions have been published. For any positive integers $N$ and $\tilde{N}$ with $N \leq \tilde{N}$, J. Kovačević and Sweldens [7] constructed an interpolatory mask satisfying sum rules of order $\tilde{N}$ and its dual mask satisfying sum rules of order $N$. In [2], H. Ji et al proposed a convolution method to construct refinable functions of arbitrary regularity which are dual to an interpolatory scaling function. For an interpolatory mask, B. Han [1] has provided an CBC algorithm to construct the dual masks which satisfy sum rules of any given order. However, up to now, there is still no simple method to construct the two-dimensional biorthogonal wavelets from the interpolatory refinable function. The method provided by H. Ji et al. [2] needs not only the Gram-Schmidt processing but also the computation of the inverse of a Laurent matrix. In this paper, we provide a method for constructing the two-dimensional biorthogonal wavelets from the interpolatory refinable function. The wavelet masks are given explicitly.

The two-dimensional biorthogonal wavelet system is introduced in Section 2. In Section 3, formulas are provided for constructing biorthogonal wavelet masks from two-dimensional interpolatory function. Example is also given to demonstrate this method. Finally, the conclusion is given in Section 4.