On Approximate Linearized Triangular Decompositions

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Abstract. In this paper, we describe progress on the development of algorithms for triangular decomposition of approximate systems.
We begin with the treatment of linear, homogeneous systems with positive-dimensional solution spaces, and approximate coefficients. We use the Singular Value Decomposition to decompose such systems into a stable form, and discuss condition numbers for approximate triangular decompositions. Results from the linear case are used as the foundation of a discussion on the fully nonlinear case. We introduce linearized triangular sets, and show that we can obtain useful stability information about sets corresponding to different variable orderings. Examples are provided, experiments are described, and connections with the works of Sommese, Verschelde, and Wampler are made.

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1. Introduction

Systems of polynomial equations frequently arise in applications, and it is often of interest to determine their triangular forms. Such representations enable the expression of some of the variables as functions of the remaining ("free") variables. Already, methods exist which are designed to compute triangular sets for exact systems whose varieties are of arbitrary dimension [43, 22, 28, 42]. However, for real world problems, the systems under consideration frequently have approximate coefficients that are inferred from experimental data. This means that the stability of these triangular representations is a valid concern. To be more clear, we are not merely concerned with the sensitivity of the triangular representation to perturbations in the original data, but we also want the solution set to be stable under small changes in values taken by the free variables.

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In another paper, *On Approximate Triangular Decompositions I: Dimension Zero* [29], we gave a detailed treatment of approximate triangular decomposition for systems with finitely many roots (zero-dimensional systems). That work follows the equiprojectable decomposition presented in [9] and also makes use of the interpolation formulas recently proposed by Dahan and Schost [12]. In this second paper, we study the simplest class of positive dimensional systems: linear homogeneous systems.

It is true that Gaussian elimination, with respect to a given variable ordering, will transform any exact linear system provided as input into a triangular solved form. Solutions are parameterized by free variables which are lower in the ordering than the remaining variables. However, in the case of approximate systems, neither replacement of floating point numbers with rational numbers nor use of Gaussian elimination with full pivoting can be guaranteed to give orderings which are ideal (see the standard text [21] for a discussion of these issues). Furthermore, such methods may lead to approximate triangular representations which are practically unstable, even in the case of exact systems.

The key idea for the portion of the current paper which deals with linear, homogeneous systems, is to use stable methods from Numerical Linear Algebra. Specifically we use the Singular Value Decomposition (SVD) to determine whether a stable approximate triangular set exists for some variable ordering. We will show that such a stable set and ordering always exists, but that a given ordering may lead to an unstable representation. Furthermore, an interpretation of this set is given in terms of an exact solution to some nearby (homogeneous) system.

From there we will further explore some local structure of nonlinear problems with Linearized Approximate Triangular Decompositions. Applying results from the treatment of linear problems to the linearization of nonlinear systems, variable orderings for locally-stable approximate triangular sets can be determined. To do this, we use the homotopy continuation methods of PHCpack [41] to generate generic points on each irreducible component of a given nonlinear polynomial system [32, 34, 35, 33]. A collection of certain results on the decomposition of non-linear systems is also provided here. For example, for $n$ variables, the interpolation methods of Sommese, Verschelde, and Wampler give approximate triangular representations of the $(n - 1)$-dimensional components.

The above results, together with certain results using the equiprojectable decomposition (presented in our related work [29]), form an accessible bridge to the study of the fully non-linear case. This will be described in a forthcoming work.

2. Approximate Triangular Sets for Positive Dimensional Linear Systems

In this section we discuss triangular representations for linear systems. The foundation of our treatment is the Singular Value Decomposition, and techniques that have now become standard in Numerical Linear Algebra [21, 39].