Chapter 4

Pseudo-differential Operators on $\mathbb{T}^n$

Pseudo-differential operators on the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, or the periodic pseudo-differential operators, are studied next. The presentation is written in a way for a reader to be able to compare and to contrast it to the general theory of pseudo-differential operators on the Euclidean space from Chapter 2.

However, while in $\mathbb{R}^n$ in Chapter 2 we aimed at avoiding technicalities by restricting ourselves to symbols of type $S^{m}_{0,0}$, here we will also discuss symbols of types $S^{m}_{\rho,\delta}$. One reason for this is that no comprehensive treatment of operators on $\mathbb{T}^n$ seems to be available in the literature so we may treat a more general situation also to exhibit the dependence of some results on the values of parameters $\rho$ and $\delta$.

In Chapter 3 a sound basis for the development of the theory of periodic pseudo-differential operator was founded, where the pieces of information have been known practically for decades, and in some cases, for centuries; however, these fragments of wisdom have been scattered widely apart in the field of mathematics.

We will see how closely periodic pseudo-differential operators are tied to the general pseudo-differential operators, the main difference actually being that in the periodic case, the theory appears to be more crystallised. This is definitely good news for those who want to grasp the ideas of any pseudo-differential theory.

In 1979 Agranovich [3] proposed, crediting L.R. Volevich, a global definition of pseudo-differential operators on the unit circle $\mathbb{S}^1$, called the periodic pseudo-differential operators. Of course, the definition was readily generalisable for any torus $\mathbb{T}^n$. Due to the group structure of $\mathbb{T}^n$, by exploiting the Fourier series representation these new operators admitted globally defined symbols instead of mere local analysis. We also note here that a similar representation of operators has been already used by Petrovski in [86] in the analysis of the Cauchy problem for systems of partial differential equations.

It is a non-trivial fact, however, that the definitions of pseudo-differential operators on a torus given by Agranovich and Hörmander are equivalent. Agranovich proved this in [4] in the special case of classical operators, and later without some details in [5] in the case of the Hörmander $(1,0)$-operators. Another treatise of the classical operators was presented in [103]. A complete proof was provided by
McLean [76] for all the Hörmander \((\rho, \delta)\)-classes. McLean proved equivalence of the global and local definitions by directly studying charts of the tori. Another proof of this type was given in [79] for the \((1,0)\)-class. In the sequel, we give one more approach, based on extension and periodisation techniques, providing the equality of \((\rho, \delta)\)-symbol classes (Corollary 4.6.13), also yielding an explicit relation between operators (Theorem 4.6.12).

Periodic integral operators are a major source of applications for the periodic pseudo-differential operator theory. Unfortunately, there is not much room for discussing periodic integral operators here, except for an application in Section 4.11. Important further results on this subject are fast methods of solving periodic integral equations presented in [142] and [102], which certainly is recommended for further reading on these topic. Other applications to the numerical analysis of periodic equations in mechanics and aerodynamics can be found in, e.g., [144, 145]. Numerical aspects of Fourier transforms on general compact groups can be found in, e.g., [75].

From the point of view of these applications, a theory of pseudo-differential operators expressed in terms of Fourier coefficients and discrete operations is appealing. Periodic pseudo-differential operators were briefly considered, e.g., in [34], and certain aspects studied in [6, 7], [142], [102] and [138]. We note that analysis of vector fields has an obvious embedding into the theory of pseudo-differential operators on the torus, and thus, for example, questions of global hypoellipticity and solvability of vector fields (e.g., [42], [43], [44], [13], etc.) obtain a more fundamental ground. Some aspects of the analysis presented in this chapter appeared in [97, 98].

In this chapter we develop the foundations of the theory of pseudo-differential operators on the torus \(\mathbb{T}^n\). This includes toroidal quantization of operators, toroidal symbol classes, toroidal amplitudes, asymptotic expansions, symbolic calculus, boundedness on \(L^2(\mathbb{T}^n)\) and on the Sobolev spaces \(H^s(\mathbb{T}^n)\), questions of ellipticity and regularity. Section 4.11 gives an application to periodic integral equations.

In Sections 4.12 we consider toroidal wave front sets relating them to the standard Hörmander wave front sets in \(\mathbb{R}^n\). In Section 4.13 we introduce Fourier series operators and study their compositions with pseudo-differential operators in terms of the toroidal symbols. In Section 4.14 we establish the boundedness of Fourier series operators in \(L^2(\mathbb{T}^n)\) and \(H^s(\mathbb{T}^n)\), and in Section 4.15 we discuss an application to the Cauchy problem for hyperbolic partial differential equations.

Fourier series operators considered here are analogues of the Fourier integral operators on the torus and we study them in terms of the toroidal quantization. The main new difficulty here is that while pseudo-differential operators do not move the wave front sets of distributions, this is no longer the case for Fourier series operators. Thus, we are forced to make extensions of functions from the integer lattice to the Euclidean space on the frequency side in Theorem 4.13.11. However, the other composition formula in Theorem 4.13.8 is still expressed entirely in the toroidal language.