Chapter 9

Boundedness and Continuity of Convex Functions and Additive Functions

9.1 The classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$

The Theorem of Bernstein–Doetsch (cf., in particular, Corollary 6.4.1) says that if $D \subset \mathbb{R}^N$ is a convex and open set, $f : D \to \mathbb{R}$ is a convex function, $T \subset D$ is open and non-empty, and $f$ is bounded above on $T$, then $f$ is continuous in $D$. Are there other sets $T$ with this property? What are possibly weak conditions which assure the continuity of a convex function, or of an additive function? In this and in the next chapter we will deal with such questions.

In order to simplify the notation we introduce the following classes of sets (Ger-Kuczma [115]).

\[
\mathcal{A} = \{ T \subset \mathbb{R}^N | \text{ every convex function } f : D \to \mathbb{R}, \text{ where } T \subset D \subset \mathbb{R}^N \text{ and } D \text{ is convex and open, bounded above on } T \text{ is continuous in } D \},
\]

\[
\mathcal{B} = \{ T \subset \mathbb{R}^N | \text{ every additive function } f : \mathbb{R}^N \to \mathbb{R} \text{ bounded above on } T \text{ is continuous} \},
\]

\[
\mathcal{C} = \{ T \subset \mathbb{R}^N | \text{ every additive function } f : \mathbb{R}^N \to \mathbb{R} \text{ bounded on } T \text{ is continuous} \}.
\]

If we want to stress that the classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ refer to a particular space $\mathbb{R}^N$, we will write $\mathcal{A}_N, \mathcal{B}_N, \mathcal{C}_N$ instead of $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

Since every additive function is convex (cf. 5.3), it follows directly from the definition that

\[
\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}. \tag{9.1.1}
\]

In the next chapter it will be proved (Theorem 10.2.2) that actually

\[
\mathcal{A} = \mathcal{B}. \tag{9.1.2}
\]

Relation (9.1.2) implies that the class $\mathcal{A}$ is independent of the choice of the convex and open set $D$ occurring in the definition. Relation (9.1.2) implies also that there
are no special conditions guaranteeing that a set $T \in \mathcal{B}$ which would not guarantee
at the same time that also $T \in \mathcal{A}$. In the sequel of this chapter we will freely use
relation (9.1.2), deferring its proof to 10.2.

On the other hand, the inclusion $\mathcal{B} \subset \mathcal{C}$ is strict, i.e., the class $\mathcal{C}$ is strictly larger
than $\mathcal{B}$. Namely we have the following

**Theorem 9.1.1.** $\mathcal{C} \setminus \mathcal{B} \neq \emptyset$.

**Proof.** Let $f_0 : \mathbb{R}^N \to \mathbb{R}$ be a discontinuous additive function, and put

$$T = \{ x \in \mathbb{R}^N \mid f_0(x) \leq 0 \}. \quad (9.1.3)$$

Then $T \notin \mathcal{B}$, since there exists a discontinuous additive function (viz. $f_0$) bounded
above on $T$. We will show that $T \in \mathcal{C}$.

Let $f : \mathbb{R}^N \to \mathbb{R}$ be an arbitrary additive function which is bounded on $T$. Thus
there exists a constant $M > 0$ such that

$$|f(x)| \leq M \quad \text{for } x \in T. \quad (9.1.4)$$

Take an arbitrary $x_0 \in \mathbb{R}^N$. By Theorem 5.2.1

$$f(kx_0) = kf(x_0), \quad (9.1.5)$$

and

$$f_0(kx_0) = kf_0(x_0) \quad (9.1.6)$$

for every $k \in \mathbb{Z}$. By (9.1.6), depending on the sign of $f_0(x_0)$, $kx_0 \in T$ for $k = 1, 2, 3, \ldots$, or for $k = -1, -2, -3, \ldots$. Making use of (9.1.5) and (9.1.4) we get hence

$$|kf(x_0)| \leq M$$

for $k = 1, 2, 3, \ldots$, or for $k = -1, -2, -3, \ldots$. This is possible if and only if $f(x_0) = 0$.

Consequently $f(x_0) = 0$ for arbitrary $x_0 \in \mathbb{R}^N$, i.e., $f = 0$, and hence it is
continuous. Thus every additive function $f : \mathbb{R}^N \to \mathbb{R}$ bounded on $T$ is continuous,
i.e., $T \in \mathcal{C}$. Hence $T \in \mathcal{C} \setminus \mathcal{B}$. \hfill $\Box$

Let us also note the following

**Theorem 9.1.2.** Let $f : \mathbb{R}^N \to \mathbb{R}$ be an additive function bounded below on a set $T \in \mathcal{B}$. Then $f$ is continuous.

**Proof.** If $f$ is an additive function bounded below on $T$, then $-f$ is an additive
function bounded above on $T$. Since $T \in \mathcal{B}$, this implies that $-f$ is continuous, and
hence also $f$ is continuous. \hfill $\Box$