The Ablowitz–Ladik Hierarchy Revisited

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Abstract. We provide a detailed recursive construction of the Ablowitz–Ladik (AL) hierarchy and its zero-curvature formalism. The two-coefficient AL hierarchy under investigation can be considered a complexified version of the discrete nonlinear Schrödinger equation and its hierarchy of nonlinear evolution equations.

Specifically, we discuss in detail the stationary Ablowitz–Ladik formalism in connection with the underlying hyperelliptic curve and the stationary Baker–Akhiezer function and separately the corresponding time-dependent Ablowitz–Ladik formalism.

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1. Introduction

The prime example of an integrable nonlinear differential-difference system to be discussed in this paper is the Ablowitz–Ladik system,

\[\begin{align*}
-\alpha t - (1 - \alpha \beta)(\alpha^- + \alpha^+) + 2\alpha &= 0, \\
-\beta t + (1 - \alpha \beta)(\beta^- + \beta^+) - 2\beta &= 0
\end{align*}\]  

(1.1)

with \(\alpha = \alpha(n,t), \beta = \beta(n,t), (n,t) \in \mathbb{Z} \times \mathbb{R}\). Here we used the notation \(f^\pm(n) = f(n \pm 1), n \in \mathbb{Z}\), for complex-valued sequences \(f = \{f(n)\}_{n \in \mathbb{Z}}\). The system (1.1) arose in the mid-seventies when Ablowitz and Ladik, in a series of papers [3]–[6] (see also [1], [2, Sect. 3.2.2], [7, Ch. 3], [17]), used inverse scattering methods to analyze certain integrable differential-difference systems. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where \(\beta = -\pi\), and in the defocusing case, where \(\beta = \pi\), (1.1) yields the discrete analog of the

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nonlinear Schrödinger equation
\[ -i\alpha_t - (1 \pm |\alpha|^2)(\alpha^- + \alpha^+) + 2\alpha = 0. \] (1.2)

We will refer to (1.1) as the Ablowitz–Ladik system. The principal theme of this paper will be to derive a detailed recursive construction of the Ablowitz–Ladik hierarchy, a completely integrable sequence of systems of nonlinear evolution equations on the lattice \(\mathbb{Z}\) whose first nonlinear member is the Ablowitz–Ladik system (1.1). In addition, we discuss the zero-curvature formalism for the Ablowitz–Ladik (AL) hierarchy in detail.

Since the original discovery of Ablowitz and Ladik in the mid-seventies, there has been great interest in the area of integrable differential-difference equations. Two principal directions of research are responsible for this development: Originally, the development was driven by the theory of completely integrable systems and its applications to fields such as nonlinear optics, and more recently, it gained additional momentum due to its intimate connections with the theory of orthogonal polynomials. In this paper we will not discuss the connection with orthogonal polynomials (see, however, the introduction of [31]) and instead refer to the recent references [13], [20], [37], [38], [42], [43], [44], [47], [48], [49], and the literature cited therein.

The first systematic discussion of the Ablowitz–Ladik (AL) hierarchy appears to be due to Schilling [45] (cf. also [51], [55], [58]); infinitely many conservation laws are derived, for instance, by Ding, Sun, and Xu [21]; the bi-Hamiltonian structure of the AL hierarchy is considered by Ercolani and Lozano [23]; connections between the AL hierarchy and the motion of a piecewise linear curve have been established by Doliwa and Santini [22]; Bäcklund and Darboux transformations were studied by Geng [26] and Vekslerchik [56]; the Hirota bilinear formalism, AL \(\tau\)-functions, etc., were considered by Vekslerchik [55]. The initial value problem for half-infinite AL systems was discussed by Common [19], for an application of the inverse scattering method to (1.2) we refer to Vekslerchik and Konotop [57]. This just scratches the surface of these developments and the interested reader will find much more material in the references cited in these papers and the ones discussed below.

Algebro-geometric (and periodic) solutions of the AL system (1.1) have briefly been studied by Ahmad and Chowdhury [8], [9], Bogolyubov, Prikarpatskii, and Samoilenko [14], Bogolyubov and Prikarpatskii [15], Chow, Conte, and Xu [18], Geng, Dai, and Cao [27], and Vaninsky [53].

In an effort to analyze models describing oscillations in nonlinear dispersive wave systems, Miller, Ercolani, Krizevich, and Levermore [40] (see also [39]) gave a detailed analysis of algebro-geometric solutions of the AL system (1.1). Introducing

\[ U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}, \quad V(z) = \begin{pmatrix} z - 1 - \alpha\beta \alpha - \alpha^{-1}z^{-1} \\ \beta z - \beta \beta^{-1} - \beta z^{-1} \end{pmatrix} \] (1.3)

with \(z \in \mathbb{C} \setminus \{0\}\) a spectral parameter, the authors in [40] relied on the fact that the Ablowitz–Ladik system (1.1) is equivalent to the zero-curvature equations

\[ U_t + UV - V^+U = 0. \] (1.4)