Chapter 7

Mathematical Foundations of Fuzzy Logic

Our motivation to enter into the foundations of fuzzy logic is based on five aspects. First, fuzzy logic complements parity logic in a unique way and represents currently the most versatile branch of approximate and causal reasoning, in particular with respect to fuzzy cognitive maps in almost every field of psychology. Second, the space $B^l$ is of fundamental importance to fuzzy logic, because it provides the search space for optimizing fuzzy unit (fit) vectors $A = (a_1, a_2, \ldots, a_n)$ in the unit hypercube $I^n$. The point at issue is that linguistic variables mean different things to different people. Even experts differ in categorizing the values of information and control variables. This is a problem of meaning, and it is solvable in principle by subjecting fit-vectors of length $n$ to evolutionary genetic optimization, i.e. to submit them to special parity feedback machines which localize satisficing or optimal fit-vectors $A$ in $I^n$, whose artificial genotypes are $l$-dimensional bit-vectors in $B^l$ which encode these fit-vectors.

Third, fuzzy logic and parity logic link the two entropy concepts of fuzzy entropy and iso-entropy. The former is a key concept for the emergence of meaning regarding linguistic variables such as velocity with values slow, medium, or fast. Linguistic variables are not only context dependent, they are also functional in the sense that their values stand for something that signifies their meaning. Fuzzy entropy tells us how fuzzy a fuzzy set is. Optimizing fuzzy entropy, i.e. optimi-
zing the generic uncertainty of fuzzy sets amounts to optimizing fuzzy subsethood, and that in turn means getting the best adapted fuzzy in- and output sets for the architecture of control spaces and fuzzy rule banks. While crisp XOR is entropy preserving and thus the hard core of parity logic, its fuzzy counterpart, called fuzzy XOR, preserves fuzzy entropy and plays a significant role in the foundations of fuzzy logic.

Fourth, fuzzy logic and parity logic are both at the heart of computational and algorithmic compression. Parity logic serves to compress algorithmic procedures that are otherwise tedious, inefficient, and ill-suited to the computer, whereas fuzzy logic realizes data compression and granulation\(^1\). Fifth, fuzzy logic and parity logic are mutually challenging. They coexist as separate logical disciplines, but their powers may be combined most fruitfully in terms of autogenetic fuzzy logic systems. A primal thesis of Part III is that parity logic unites fuzzy logic and evolutionary computation into the cohesive framework of hypercubical calculus. To support this thesis, we have to furnish the appropriate mathematical background of the foundations of fuzzy logic.

The purpose of section 7.1 is to provide a compact survey of the unit hypercube \(I^n = [0, 1]^n\), thereby getting a crystal clear conception of fuzzy sets and equivalent concepts. Section 7.2 centers on conceptual foundations and computational procedures for fuzzy unit viz. fit-vectors, in particular Kosko’s geometric view of sets-as-points, and new aspects of fuzzy XOR. The subject of section 7.3 is then the problem of meaning and its treatment on the grounds of fuzzy entropy and subsethood. Section 7.4 presents then finally generalized fuzzy inner- and outer product operators, since they play a significant role in fuzzy logic systems, the subject of chapter 8.

As in chapter 2, we present these mathematical foundations in standard computer science nomenclature, but we will refer again to the nomenclature of APL, because APL provides \textit{hundreds} of distinct inner- and outer products. For instance, fuzzy vector-matrix multiplication, the max-min composition \(A \circ M = B = (A \vee M)\), is one out of 450 distinct inner product operators. A similar argument holds for fuzzy correlation-product encoding, the fuzzy Hebbian ma-

\(^{1}\)Take, as an example, the linguistic variable \textit{temperature} whose linguistic values are \textit{low}, \textit{moderate}, and \textit{high}. Then the value \textit{high} represents a choice of one out of three possible values, whereas 95 (degrees of Fahrenheit) would be a choice of one out of, say, 180 values. Hence, overlapping fuzzy sets granulate the universe of discourse, and hosts of data are compressed into a few terms ([ZAD94]).