Chapter 8
Stochastic Comparisons of Spacings from Heterogeneous Samples

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Abstract In this paper we review some of the recently obtained results in the area of stochastic comparisons of sample spacings when the observations are not necessarily identically distributed. A few new results on necessary and sufficient conditions for various stochastic orderings among spacings are also given. The paper is concluded with some examples and applications.

8.1 Introduction

Spacings are of great interest in many areas of statistics, in particular, in the characterizations of distributions, goodness-of-fit tests, auction theory, life testing and reliability models. A large number of goodness-of-fit tests are based on functions of sample spacings (see [2, 14, 15]).

Let $X_1, \ldots, X_n$ be n nonnegative random variables. The random variables $D_{i:n} = X_{i:n} - X_{i-1:n}$ and $D^*_{i:n} = (n-i+1)D_{i:n}$, $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$, are respectively called simple spacings and normalized spacings. In the reliability context they correspond to times elapsed between successive failures of components in a system. In stochastic auction theory, $D_{n:n}$ and $D_{2:n}$ are of particular interest, which represent auction rents in buyer's auction and reverse auction in the second-price business auction (see [37]). It is well known that the normalized spacings of a random sample from an exponential distribution are independent and identically distributed (i.i.d.) random variables having the same exponential distribution. Such a characterization may not hold for other distributions or when the observations are not independent and identically distributed.

In many cases, the observations are independent but not identically distributed and we call them as heterogeneous samples. The study of heterogeneous samples is of great interest in many areas. For examples, in engineering, a complex engineering

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system is often composed of many different types of electrical components. Investigating the reliability of such system is relying on heterogeneous samples, which are the failure times of electrical components collected from experiments. Accordingly, the topic of heterogenous samples plays an important role. However, such a study is often challenging as the distribution theory of spacings when the observations are not i.i.d. is often complicated. A powerful tool to investigate the stochastic properties of spacings of heterogeneous samples is *stochastic orders*, which is a widely studied concept in probability and statistics.

In this paper, we review some recent results on stochastic orderings between spacings of heterogeneous samples, and some new results and applications are presented as well. The other interesting topic not discussed in our paper is that of dependence among spacings. Interested readers may refer to recent papers by [4, 5, 13, 16, 30] and references therein on this topic.

8.2 Stochastic Orders

Let X and Y be two random variables with distributions F and G, and survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$. If X is less likely than Y to take on large values, then, intuitively, the survival function of X will be smaller than the survival function of Y at any fixed point. This leads to the *usual stochastic order*.

Definition 8.1. X is said to be smaller than Y in the usual stochastic order, denoted by $X \le_{st} Y$, if $\overline{F}(x) \le \overline{G}(x)$, or equivalently, $F(x) \ge G(x)$.

Suppose that X and Y are life lengths of two electronic components and satisfy $X \le_{st} Y$. If both components are observed to be alive at time $t > 0$, one might conjecture that the residual lives would also be stochastically ordered. However, such a result does not hold (cf. [24, 26])! Hence, a stronger concept than usual stochastic order is needed. The following order is motivated by the fact that the above conjecture is false.

Definition 8.2. X is said to be smaller than Y in the hazard rate order, denoted by $X \le_{hr} Y$, if

$$P(X > x + t | X > t) \le P(Y > x + t | Y > t), \quad \text{for all } x \ge 0 \text{ and all } t.$$

Now, suppose that two components X and Y failed before observation time $t > 0$. If $X \le_{st} Y$, is it necessarily true that the 'inactive time' $[t - X | X \le t]$ of X is stochastically larger that the 'inactive time' $[t - Y | Y \le t]$ of Y? The answer is negative. The following definition is proposed to resolve this question.

Definition 8.3. X is said to be smaller than Y in the reverse hazard rate order, denoted by $X \le_{rh} Y$, if

$$P(t - X > x | X \le t) \ge P(t - Y > x | Y \le t), \quad \text{for all } x \ge 0 \text{ and all } t.$$

An interesting order based on the mean residual life is defined as follows.