Piecewise Linear Bertrand Oligopoly

Joachim Rosenmüller

Institute of Mathematical Economics
University of Bielefeld

Abstract

We describe a modell of price competition between firms with piecewise linear cost functions. Thus, we consider "Bertrand oligopoly", an n-person noncooperative game in which players choose prices and the market, reflected by a decreasing demand function, reacts discontinuously as total demand concentrates on those firms that offer minimal prices. Firms do not have to be identical. But a notion of similarity between firms is necessary in order to prove the existence of a Nash (-Bertrand) equilibrium. Here we are only interested in an equilibrium involving all firms – the case of subgroups with "similar" members deserves an additional study.

1 Bertrand Oligopoly

Within this paper we discuss the existence of equilibria within a certain type of Bertrand Oligolpoly. [1] The main feature is the structure of the cost functions of the firm, these are supposed to be piecewise linear and convex. Such cost functions appear naturally in the context of network flow structures, where flows passing through capacity limited nodes and edges generate costs depending on the choice of edges as well. We think of such kind of flow as electricity or data material on an electronic net. Routing the flow optimally (cost minimizing) results in a linear programming problem, the solution of which yields a piecewise linear cost function. See [6] for a detailed model of this type.

The technique is not far away from standard procedures. However, apart from missing differentiability assumptions we also do not assume symmetric firms.

Most of the literature seems to rely on at least one of these assumptions. DASTIDAR [2] discusses the asymmetric case as well (assuming that costfunctions are twice differentiable), however the assumptions imposed on the model vary during the presentation. HOERNIG [4] constructs in addition to the continuum of pure equilibria existing a host of mixed ones. See also MASKIN [5] for mixed equilibria. Symmetry is also assumed in

HEHENKAMP–LEININGER [3], who discuss evolutionary Bertrand equilibria.

It would seem that none of the properties derived in the context of this literature suffers when differentiability is sacrificed and firms are just "similar".

As frequently, it is assumed that firms have a limited capacity of production. Yet they are supposed to meet market demand at the level required. The game in which firms may plan to sell less than required has different strategies and payoffs. Yet it seems that the type of equilibrium exhibited would constitute an equilibrium in the extended game as well. Within our present framework, we will not attend to this question.

The model is specified essentially by a set of piecewise linear costfunctions for the firms and a demand function of the market. We specify this data as follows.

For any nonnegative convex, monotone function D on the reals we denote by D' the derivative of a linear support function of D at t. This derivative is unique up to at most countably many points.

A decreasing function is *slowly decreasing* if it does not decrease faster than $1/t$, i.e., if $\frac{D(t)}{t} \geq -D'(t)$ holds true for all t in the domain of definition. Economically this reflects nonneagtive marginal expenditure.

Given positive real numbers d_0 and p_0, we call a function

$$D \ : \ [0, p_0] \to [0, d_0]$$

a ***demand function*** if it is continuous at 0, convex, and slowly decreasing. A demand function is hence continuous and differentiable with the exception of at most countably many points.

On the other hand, let for $K \in \mathbb{N}$

$$\boldsymbol{C}^{(0)} \ := \ (\boldsymbol{A}^{(0)}, \boldsymbol{B}^{(0)}) \in \mathbb{R}^{2K} \tag{1.1}$$

be such that $\boldsymbol{A}^{(0)} = (A_k^{(0)})_{k=0,\dots K}$ and $\boldsymbol{B}^{(0)} = (B_k^{(0)})_{k=0,\dots K}$ are real numbers *strictly increasing* in k and satisfy $A_0^{(0)} = 0,\ B_0^{(0)} = 0$. We put

$$\Delta_0^{(0)} \ := \ 0, \ \Delta_k^{(0)} \ := \ \frac{\Delta B_k^{(0)}}{\Delta A_k^{(0)}} \ := \ \frac{B_k^{(0)} - B_{k-1}^{(0)}}{A_k^{(0)} - A_{k-1}^{(0)}}. \tag{1.2}$$

We assume that $\Delta_k^{(0)}$ is as well *strictly increasing* in k and satisfies

$$\Delta_K = d_0, \ A_K \Delta_K - B_K \leq p_0. \tag{1.3}$$