Continuous market models

In this chapter we present the theory of derivative pricing and hedging for continuous-time diffusion models. As in the discrete-time case, the concept of martingale measure plays a central role: we prove that any equivalent martingale measure (EMM) is associated to a market price of risk and determines a risk-neutral price for derivatives, that avoids the introduction of arbitrage opportunities. In this setting we generalize the theory in discrete time of Chapter 2 and extend the Markovian formulation of Chapter 7, based upon parabolic equations.

Our presentation follows essentially the probabilistic approach introduced in the papers by Harrison and Kreps [163], Harrison and Pliska [164]. In the first two paragraphs we give the theoretical results on the change of probability measure and on the representation of Brownian martingales. Then, we introduce the market models in continuous time and we study the existence of an EMM and its relation with the absence of arbitrage opportunities. At first we discuss pricing and hedging of options in a general framework; afterwards we treat the Markovian case that is based upon the parabolic PDE theory developed in the previous chapters: this case is particularly significant, since it allows the use of efficient numerical methods to determine the price and the hedging strategy of a derivative. We next give a coincise description of the well-known technique of the change of numeraire: in particular, we examine some remarkable applications to the fixed-income markets and prove a quite general pricing formula.

10.1 Change of measure

10.1.1 Exponential martingales

We consider a $d$-dimensional Brownian motion $(W_t)_{t \in [0,T]}$ on the space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Let $\lambda \in L^2_{loc}$ be a $d$-dimensional process: we define the ex-
Ponential martingale associated to $\lambda$ (cf. Example 5.12) as

$$Z^\lambda_t = \exp \left( - \int_0^t \lambda_s \cdot dW_s - \frac{1}{2} \int_0^t |\lambda_s|^2 ds \right), \quad t \in [0, T]. \tag{10.1}$$

We recall that the symbol “$\cdot$” denotes the scalar product in $\mathbb{R}^d$. By the Itô formula we have

$$dZ^\lambda_t = -Z^\lambda_t \lambda_t \cdot dW_t, \tag{10.2}$$

so that $Z^\lambda$ is a local martingale. Since $Z^\lambda$ is positive, by Proposition 4.40 it is also a super-martingale:

$$E \left[ Z^\lambda_t \right] \leq E \left[ Z^\lambda_0 \right] = 1, \quad t \in [0, T],$$

and $(Z^\lambda_t)_{t \in [0, T]}$ is a strict martingale if and only if $E \left[ Z^\lambda_T \right] = 1$.

**Lemma 10.1** If there exists a constant $C$ such that

$$\int_0^T \|\lambda_t\|^2 dt \leq C \quad a.s. \tag{10.3}$$

then $Z^\lambda$ in (10.1) is a martingale such that

$$E \left[ \sup_{0 \leq t \leq T} (Z^\lambda_t)^p \right] < \infty, \quad p \geq 1. \tag{10.4}$$

In particular $Z^\lambda \in \mathbb{L}^p(\Omega, P)$ for every $p \geq 1$.

**Proof.** We put

$$\hat{Z}_T = \sup_{0 \leq t \leq T} Z^\lambda_t.$$  

For every $\zeta > 0$, we have

$$P \left( \hat{Z}_T \geq \zeta \right) \leq P \left( \sup_{0 \leq t \leq T} \exp \left( - \int_0^t \lambda_s \cdot dW_s \right) \geq \zeta \right)$$

$$= P \left( \sup_{0 \leq t \leq T} \left( - \int_0^t \lambda_s \cdot dW_s \right) \geq \log \zeta \right) \leq c_1 e^{-c_2 (\log \zeta)^2}.$$

(by Corollary 9.31, using condition (10.3) with $c_1, c_2$ positive constants)

$$\leq c_1 e^{-c_2 (\log \zeta)^2}.$$

Then, by Proposition A.56 we have

$$E \left[ \hat{Z}_T^p \right] = p \int_0^\infty \zeta^{p-1} P \left( \hat{Z}_T \geq \zeta \right) d\zeta < \infty.$$

In particular for $p = 2$ we have that $\lambda Z^\lambda \in \mathbb{L}^2$ and so, by (10.2), that $Z^\lambda$ is a martingale. □