Chapter 4
Covariant measurements and optimality

4.1. Parametric symmetry groups 
and covariant measurements

All symmetry groups considered in the previous chapter were parametric 
groups (Lie groups) of transformations. This means that a parametric set 
\( \Theta = \{\theta\} \), i.e., a continuous manifold in a finite-dimensional space is 
given and elements of the group \( G = \{g\} \) act as continuous one-to-one 
mappings of the set \( \Theta \) onto itself, \( g : \theta \to g\theta \). Moreover the group \( G \) is 
itself parametrized in such a way that the group product \( g_1g_2 \) is at least 
locally continuous in \( g_1, g_2 \).

The examples are the additive group \( \mathbb{R} \) considered as the shift group for 
the real line \( \Theta = \mathbb{R} \), the group \( T \) of shifts (mod \( 2\pi \)) of the interval \( \Theta = [0, 2\pi) \) and the rotation group in the three dimensional Euclidean space 
\( \mathbb{R}^3 \). In the last case it is natural to consider the action of the group only 
on directions, i.e., unit vectors in \( \mathbb{R}^3 \). Then the rotation group becomes a 
group of transformations of the unit sphere \( \Theta = S^2 \).

Let \( G \) be a parametric group of transformations of a set \( \Theta \) and \( g \to V_g \) 
be a (continuous) projective unitary representation of \( G \) in a Hilbert space 
\( \mathcal{H} \). Let \( M(d\theta) \) be a measurement with values in \( \Theta \), i.e., a resolution 
of identity in \( \mathcal{H} \) on the \( \sigma \)-field \( \mathcal{A} (\Theta) \) of Borel subsets of \( \Theta \). The 
measurement \( M(d\theta) \) is covariant with respect to representation \( g \to V_g \) 
if

\[
V_g^* M(B) V_g = M(B_g^{-1}), \quad g \in G, \tag{4.1.1}
\]

for any \( B \in \mathcal{A} (\Theta) \), where

\[
B_g = \{\theta : \theta = g\theta', \theta' \in B\}
\]

is the image of the set \( B \) under the transformation \( g \). The notion of 
covariant measurement was introduced in the previous chapter for con-

\[1\) Since \( \Theta \) is a continuous manifold it is itself a Borel subset in a finite-dimensional space.
crete symmetry groups. Here we wish to study it from a general point of view.

The importance of this notion for quantum theory lies in the fact that it establishes a correspondence between physical parameters and certain classes of quantum measurements. Indeed, assume that $\theta$ is a parameter (in general, multidimensional) describing some aspects of the preparation procedure and $S$ the basic state corresponding to the value $\theta_0$. Then the transformation $g$ results in the preparation of the new state $S_{\theta} = V_g S V_g^*$, where $\theta = g \theta_0$. If the measurement $M(d\hat{\theta})$ is then made, the probability distribution of the results $\hat{\theta}$ of the measurement will be

$$\Pr\{\hat{\theta} \in B | \theta\} = \text{Tr} S_{\theta} M(B), \quad B \in \mathcal{A}(\Theta).$$

If $M(d\hat{\theta})$ possesses the covariance property (4.1.1), then

$$\text{Tr} S_{\theta} M(B) = \text{Tr} S V_g^* M(B) V_g = \text{Tr} S M(B_{g^{-1}})$$

whence replacing $B$ by $B_g$

$$\Pr\{\hat{\theta} \in B_g | \theta_0\} = \Pr\{\hat{\theta} \in B | \theta_0\}.$$

Thus the change of the value of $\theta$ is properly reflected by the change in the resulting probability distribution, and therefore any resolution of identity $M(d\hat{\theta})$ satisfying (4.1.1) corresponds to a theoretically admissible measurement of the parameter $\theta$.

A mathematical problem which naturally arises is to describe all covariant measurements of the given parameter $\theta$. We shall give its general solution in Section 4.2 and 4.8. Then we shall look for “optimal” measurements having the best theoretically possible accuracy among all covariant measurements of the parameter $\theta$. In this way we shall find that the “canonical” measurements for various parameters introduced in the previous chapter are just the typical representatives in the family of the optimal covariant measurements.

We shall need some general knowledge of parametric groups of transformations. The group $G$ acts transitively on $\Theta$ if any point $\theta_0$ can be transformed into any other point $\theta$ by some $g \in G$. In what follows we assume that a parametric group $G$ acts transitively on $\Theta$. Then the continuous mapping $g \rightarrow g \theta_0 = \theta(g)$ maps $G$ onto the whole $\Theta$. This mapping is one-to-one if and only if the stationary subgroup $G_0$ of transformations leaving the point $\theta_0$ invariant is trivial, i.e., reduces to the identical transformation. For an example consider the shift group of $\mathbb{R}$ and fix a point $\theta_0 \in \mathbb{R}$. Any point $\theta \in \mathbb{R}$ can be obtained from $\theta_0$ by a shift: $\theta = \theta_0 + x$. The mapping $x \rightarrow \theta_0 + x = \theta(x)$ is obviously one-to-one. The same is