Chapter 5
Markov property of Brownian motion

This chapter deals with the Markov and strong Markov properties of the Brownian motion and to several their interesting consequences.

Sections 5.1 and 5.2 include some needed preliminaries on filtrations and stopping times. We prove the useful result that if \( \tau \) is a stopping time, then \( B(t + \tau) - B(\tau) \) is still a Brownian motion. Markov and strong Markov properties are proved in Section 5.3 together with some of their consequences.

In Section 5.4 we give several applications to the Cauchy problem for the heat equation on \([0, +\infty)\) equipped with a Dirichlet, Neumann or Ventzell boundary condition at 0.

Finally, Section 5.5 is devoted to some fine properties of the zeros of the Brownian motion.

5.1. Filtrations

We are given a real Brownian motion \( B \) on a probability space \((\Omega, \mathcal{F}, P)\). Let us introduce the natural filtration of \( B \). For any \( n \in \mathbb{N}, 0 \leq t_1 < \cdots < t_n, A \in \mathcal{B}(\mathbb{R}^n) \) we consider the cylindrical set

\[
C_{t_1, \ldots, t_n; A} = \{ \omega \in \Omega : (B(t_1)(\omega), \ldots, B(t_n)(\omega)) \in A \}.
\]

Then for any \( t \geq 0 \) we denote by \( \mathcal{C}_t \) the family of all cylindrical sets \( C_{t_1, \ldots, t_n; A} \) with \( t_n \leq t \) and by \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by \( \mathcal{C}_t \).

It is clear that \((\mathcal{F}_t)_{t \geq 0}\) is an increasing family of \( \sigma \)-algebras and that \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \). \((\mathcal{F}_t)_{t \geq 0}\) is called the natural filtration of \( B \). One also says that \( \mathcal{F}_t \) contains the story of \( B \) up to \( t \).

The following useful lemma shows that independence of \( B(t) - B(s) \) by \( B(s) \) extends to independence of \( B(t) - B(s) \) by any \( \mathcal{F}_s \)-measurable random variable.

**Lemma 5.1.** Let \( t > s > 0 \), and let \( X \) be a real random variable \( \mathcal{F}_s \)-measurable. Then \( B(t) - B(s) \) and \( X \) are independent.
Proof. We have to show that for any \( I \in \mathcal{B}(\mathbb{R}) \), and any \( A \in \mathcal{F}_s \) it results
\[
\mathbb{P}[(B(t) - B(s) \in I) \cap A] = \mathbb{P}(B(t) - B(s) \in I) \mathbb{P}(A). \tag{5.1}
\]
By a straightforward application of Dynkin’s theorem (see Appendix A), it is enough to prove that
\[
\mathbb{P}[(B(t) - B(s) \in I) \cap \mathcal{C}_s,\ldots, J_n; A] = \mathbb{P}(B(t) - B(s) \in I) \mathbb{P}(\mathcal{C}_s,\ldots, J_n; A), \tag{5.2}
\]
for any cylindrical set of \( \mathcal{F}_s, \mathcal{C}_s,\ldots, J_n; A \), with \( 0 < s_1 < \cdots < s_n \leq s \), \( J_n \in \mathcal{B}(\mathbb{R}^n) \).

Now we recall that by Proposition 4.3 the \( \mathbb{R}^{n+2} \)-valued random variable
\[
(B(s_1), \ldots, B(s_n), B(s), B(t))
\]
is Gaussian and possesses a density \( \rho(\eta_1, \ldots, \eta_n, \xi_1, \xi_2) \) with respect to the Lebesgue measure in \( \mathbb{R}^{n+2} \) given by
\[
\rho((\eta_1, \ldots, \eta_n, \xi_1, \xi_2)) = (2\pi)^{-(n+2)/2} \left( s_n - s_{n-1}, \ldots, s_1 - s, t - s \right)^{-1/2} \tag{5.3}
\]
\[
\times e^{-\frac{\eta_1^2}{2n} - \frac{(\eta_2 - \eta_1)^2}{2(\eta_2 - s_{n-1})} - \frac{(\eta_3 - \eta_2)^2}{2(\eta_3 - s_{n-2})} - \frac{(\eta_4 - \eta_3)^2}{2(\eta_4 - s_{n-3})} - \frac{(\eta_n - \eta_{n-1})^2}{2(\eta_n - s)} - \frac{(\xi_1 - \eta_n)^2}{2(\xi_1 - s)} - \frac{(\xi_2 - \eta_{n-1})^2}{2(\xi_2 - s_{n-1})}} d\eta_1 \cdots d\eta_n d\xi_1 d\xi_2.
\]
Now, checking (5.2) is a simple computation which is left to the reader. \( \square \)

5.1.1. Some properties of the filtration \( (\mathcal{F}_t) \)

We first notice that, as easily checked, for all \( t > 0 \) \( \mathcal{F}_t \) coincides with the \( \sigma \)-algebra generated by all cylindrical sets of the form
\[
C_{t_1,\ldots,t_n}; A_n, \quad 0 < t_1 < \cdots < t_n < t, \quad A_n \in \mathcal{B}(\mathbb{R}^n), \quad n \in \mathbb{N}.
\]
We say that filtration \( (\mathcal{F}_t)_{t \geq 0} \) is left continuous.

For any \( t \geq 0 \) we define the right limit at \( t \mathcal{F}_t^+ \) of \( (\mathcal{F}_t)_{t \geq 0} \) setting
\[
\mathcal{F}_t^+ := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}.
\]

Exercise 5.2. Consider the set
\[
D := \{ \omega \in \Omega : B(\cdot)(\omega) \text{ is differentiable at } 0 \}.
\]
Show that \( D \) is non empty and belong to \( \mathcal{F}_{0+} \).