Chapter 6
Spectral optimization problems in $\mathbb{R}^d$

6.1. Optimal sets for the $k$th eigenvalue of the Dirichlet Laplacian

The aim of this section is to study the optimal sets for functionals depending on the eigenvalues of the Dirichlet Laplacian. A typical example is the model problem

$$\min \left\{ \lambda_k(\Omega) : \Omega \subset \mathbb{R}^d, \Omega \text{ quasi-open , } |\Omega| = c \right\}, \quad (6.1)$$

where $c > 0$ is a given constant. The existence of an optimal set for the problem (6.1) was proved recently by Bucur (see [20]) and by Mazzoleni and Pratelli (see [81]) two completely different techniques.

In [81] the authors reason on the minimizing sequence, proving that by modifying each set in an appropriate way, one can find another minimizing sequence composed of uniformly bounded sets. At this point the classical Buttazzo-Dal Maso theorem (see Theorem 2.82) can be applied.

The argument in [20] is based on a concentration-compactness principle in combination with an induction on $k$. The boundedness of the optimal set is fundamental for this argument and is obtained using the notion of energy subsolutions. We note that this technique can easily be generalized and applied to other situations (optimization of potentials, capacitary measures, etc). The price to pay is the fact that some minor restrictions are needed on the spectral functional. More precisely, for the penalized version of the problem it is required that the spectral functional is Lipschitz with respect to the eigenvalues involved, while in [81] was shown in the case of domains this assumption can be dropped.

We note that by a simple rescaling argument (see Remark 6.3), the problem (6.1) is equivalent to

$$\min \left\{ \lambda_k(\Omega) + m|\Omega| : \Omega \subset \mathbb{R}^d, \Omega \text{ quasi-open } \right\}, \quad (6.2)$$
for some positive constant $m$, which we call Largange multiplier. For general spectral functionals of the form

$$\mathcal{F}(\Omega) = F(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)),$$

the Largange multiplier problem is easier to threat, due to the fact that any quasi-open set can be used to test (6.2). The connection between the optimization problem at fixed measure and the penalized one is, in general, a technically difficult question; further complications appear if we optimize under additional geometric constraints.

Our first result in this section concerns the existence of an optimal set for the problem (6.2). Our result is more general and concerns shape optimization problems of the form

$$\min \left\{ F(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)) + |\Omega| : \Omega \subset D, \ \Omega \text{ quasi-open} \right\}, \ (6.3)$$

where $k_1, \ldots, k_p \in \mathbb{N}$ and the function $F : \mathbb{R}^p \to \mathbb{R}$ satisfies some mild monotonicity and continuity assumptions. More precisely we work with functionals satisfying the following definition.

**Definition 6.1.** We will say that the function $F : \mathbb{R}^p \to \mathbb{R}$ is:

- **increasing**, if for each $x \geq y \in \mathbb{R}^p$, we have that $F(x) \geq F(y)^1$;
- **diverging at infinity**, if $\lim_{x \to +\infty} F(x) = +\infty$. More precisely, $\lim_{n \to \infty} F(x_n) \to +\infty$, for every sequence $x_n = (x_{1n}, \ldots, x_{pn}) \in \mathbb{R}^p$ such that $\lim_{n \to \infty} x_{in} = +\infty$, for every $i = 1, \ldots, p$.
- **increasing with growth at least $a > 0$**, if $F$ is increasing and the constant $a > 0$ is such that, for every $x \geq y$, we have
  $$F(x) - F(y) \geq a|x - y|.$$

**Theorem 6.2.** Consider the set $\{k_1, \ldots, k_p\} \subset \mathbb{N}$ and let $F : \mathbb{R}^k \to \mathbb{R}$ be an increasing and locally Lipschitz function diverging at infinity. Then there exists a quasi-open set, solution of the problem (6.3). Moreover, under the above assumptions on $F$, every solution of (6.3) is a bounded set of finite perimeter.

If, furthermore, the function $F$ is increasing with growth rate at least $a > 0$, then for every optimal set $\Omega$, there are orthonormal and Lipschitz continuous eigenfunctions $u_{k_1}, \ldots, u_{k_p} \in H_0^1(\Omega)$, corresponding to the eigenvalues $\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)$.

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1 We say that $x = (x_1, \ldots, x_p) \geq y = (y_1, \ldots, y_p)$, if $x_j \geq y_j$ for every $j = 1, \ldots, p$. 