Linear elasticity solutions

The equations of linear elasticity are derived in chapters 1 and 2, and can be divided into three groups: the equilibrium equations, the strain displacement equations, and the constitutive laws. Figure 3.1 shows these three groups of equations in a block diagram.

The equilibrium equations express the equilibrium conditions for a differential element of the body in terms of the stress field. These equilibrium conditions are a direct consequence Newton’s laws applied to a differential element of the deformable body. They consists of the three partial differential equations of equilibrium, eqs. (1.4).

The strain-displacement equations, also called the kinematic equations, describe the deformation of the body without reference to the forces that create the deformation. The strain components are defined based on a purely kinematic description of the deformed and undeformed configurations of the solid. The strain-displacement equations consists of the six partial differential equations relating the strain components to the displacement components, eqs. (1.63) and (1.71).
The constitutive laws describe the behavior of materials under load. More specifically, they take the form of relationships linking the stress and strain components at a point. Constitutive laws are rooted in material science and express an approximation to the observed behavior of actual materials. For Hooke’s law, they consist of six algebraic equations, eqs. (2.4) and (2.9).

A total of 15 equations of linear elasticity are obtained. Given the proper boundary conditions, these 15 equations can be solved to obtain the following 15 unknowns: the three components of the displacement vector, the six components of the strain tensor, and the six components of the stress tensor.

In addition, the six partial differential strain compatibility equations, eqs. (1.106), impose certain continuity conditions on the displacement components that may arise from a state of strain. While these compatibility equations are not part of the basic 15 equations of elasticity, their use may be a critical element of any solution procedure. In this chapter, solutions of this set of equations will be presented for very simple problems. Indeed, exact solutions for realistic problems are very difficult to obtain in general.

### 3.1 Solution procedures

The linear equations of elasticity form a set of coupled partial differential equations that are elegantly simple but like most partial differential equations, are often quite difficult to solve for realistic problems. Considerable simplification can be achieved when the general, three-dimensional formulation is reduced to a two-dimensional formulation by assuming the problem to be either plane stress or plane strain, as discussed in sections 1.3 or 1.6, respectively. Further simplification can be achieved for problems presenting specific symmetries. For example, the governing equations for two-dimensional problems featuring cylindrical symmetry reduce to ordinary differential equations. It is often necessary, however, to reformulate the elasticity equations in cylindrical or spherical coordinates to take advantage of specific symmetries or easily impose boundary conditions.

Three approaches are available for the solution of elasticity problems.

1. **Displacement formulations**: the objective is to derive three equations for the three unknown displacement components.
2. **Stress formulations**: the objective is to solve for the state of stress in the body. This means that six equations are required for the six stress components.
3. **Semi-inverse approaches**: assumptions are made to solve the problem for a subset of the variables. With that solution at hand, the remaining equations of the problem are solved. If all equations can be exactly satisfied, an exact solution is obtained and the initial assumptions are validated.

For all three approaches, dimensional reduction is often performed first. Under specific conditions, the initial three-dimensional problem can be reduced to a two- and sometimes one-dimensional problem, considerably easing the solution process. Examples of these various approaches are given in the following sections.