Chapter 23

Stability and Vibrations

Some problems of stability of equilibrium and motion have been put in evidence, in an incipient form, in the frame of Newtonian mechanics, in several of the preceding chapters; but these problems need a more profound study, with a general character, in the frame of Lagrangian or Hamiltonian mechanics. The motion of a mechanical system can take the form of linear or even non-linear vibrations; the corresponding mechanical phenomenon needs a thorough study in a general form too.

After a study of the stability of discrete mechanical systems, one considers the vibrations with a deterministic character of these systems. The corresponding mechanical phenomena are linked between them, the vibrations around a stable position of equilibrium playing a very important rôle.

23.1 Stability of Mechanical Systems

In the study of the behaviour of a mechanical system , at rest with respect to an inertial frame of reference, appears also the notion of stability, at a small change of its configuration of equilibrium. In a qualitative definition, if the mechanical system returns to its initial configuration, then the position of equilibrium is stable; otherwise, it is instable. A first study for a particle has been made in Sects. 4.1.1.7 and in 7.2.3.

The respective problems are extended to the motion of a discrete mechanical system, case in which intervene also the velocities of the particles at an initial state, which must be maintained in certain limits to can speak about the stability of the motion. A study in the frame of Hamiltonian mechanics, in the phase space , is interesting in this order of ideas.

We will make, in what follows, a study of the stability of equilibrium and motion, followed by various applications.

23.1.1 Stability of Equilibrium

After some introductory notions, we present the Lagrange–Dirichlet theorem and the theorems of Lyapunov and Chetaev. Thus, important criteria concerning the stability of equilibrium are given. We mention a detailed study of the linear systems too (Chetaev, N.G., 1963; Lyapunov, A.M., 1949).
23.1.1.1 Introductory Notions

To fix the ideas, we start from a very simple particular case, that is the case of a particle $P$ of weight $G$, constrained to stay on a fixed smooth circle, situated in a vertical plane (Fig. 23.1) (see Sects. 4.1.1.5 and 4.1.1.7 too); we have to do with an ideal constraint. The positions of equilibrium are $P_1$ and $P_2$. The problem is put analogously in case of the mathematical pendulum (see Sect. 7.1.3.1).

By a small perturbation of the position of equilibrium $P_2$, that is by a displacement of the particle $P$ in a position sufficiently close to $P_2$ and imparting to it an initial velocity of sufficiently small intensity, one obtains a motion in which the particle $P$:

(i) remains in an arbitrarily small neighbourhood of the position of equilibrium $P_2$;
(ii) the kinetic energy (hence, the modulus of the velocity of the particle) remains inferior to an as small as we wish quantity. The properties (i) and (ii) characterize the stability of the position of equilibrium $P_2$.

These ideas can be correspondingly generalized for a discrete mechanical system $\mathcal{S}$ (we will consider such systems in what follows), the position of which is specified by a representative point $P$ in the configuration space $\mathcal{A}$ or in the phase space $\mathcal{F}_2$. Thus, we say that a position of equilibrium of a discrete mechanical system $\mathcal{S}$ is stable if, after a sufficiently small perturbation of the corresponding representative point $P$, with sufficiently small arbitrary velocities, so that the magnitude of the kinetic energy of the mechanical system $\mathcal{S}$ be sufficiently small, it results a motion in which the representative point $P$ occupies a position in a as small as we wish neighbourhood with respect to the position of equilibrium, while the magnitude of the kinetic energy (hence, the magnitude of the velocities of all particles (or of all rigid solids which form the system) of the mechanical system $\mathcal{S}$) remains inferior to an arbitrary small limit.

Otherwise, if the representative point $P$ leaves a neighbourhood of the position of equilibrium or if the kinetic energy of the mechanical system $\mathcal{S}$ is greater than a positive number which does not depend on the initial conditions (hence, the magnitude of the velocities of some of the particles have this property too), then the positions of equilibrium is instable.