Chapter 24

Dynamical Systems. Catastrophes and Chaos

In 1776, ninety years after the apparition of the fundamental treatise of Newton, Laplace enounces his famous principle of the determinism, stating that: “The actual stage of the system of nature is, obviously, a consequence of that it was at the preceding moment and, if we imagine an intelligence, which – at a given moment – knows all the relations between entities of this universe, then it could establish the respective positions and the motions of all these entities, at any moment in the past or in the future” This determinism is – in fact – a mechanistic determinism. But Laplace continues: ... “there exist things which are uncertain for us, things which are more or less probable and we try to counter-balance the impossibility to know them, determining various degrees of probability”. We are thus obliged, at a certain level of knowledge, to accept also a probabilistic principle, which – by Laplace – depends on the accuracy of the instruments of measure. Hundred thirty years later, in 1903, Henri Poincaré observes that: “A very little cause, which escapes from our observation, can lead to a sensible effect and then we say that the effect is due to the chance. It can happen that small differences at the initial conditions do produce an enormous error in what will be later. The prediction becomes thus impossible and we have to do with unforeseeable phenomena”. As an example of the sensibility of the differential equations to initial conditions, E.N. Lorenz, professor of meteorology, says: “If a butterfly which stays today on a flower flaps or not its wings, that has not a great influence on the weather in the following days, but – in exchange – can have a great influence on the weather some years after.” This fact is known today as the Lorenz’s butterfly effect.

The uncertainty principle of Heisenberg according to which the position and the momentum of an elementary particle cannot be determined simultaneously with a precision as great as we wish, the Brownian motion, characterized by a great number of collisions between the particles of a very fine solid suspension in a liquid and its molecules, and many other phenomena put in evidence the necessity to introduce notions of the theory of probability as well as the aleatory variables. There appears thus the notion of chaos; and if the chaotic motions are produced in deterministic conditions, then there appears the notion of deterministic chaos, introduced forty years ago by D. Ruelle and F. Takens, by describing some phenomena of turbulent flow. The study of the causes which produce this paradoxical phenomenon introduces the notion of attractor in various forms: punctual attractor, periodic attractor and chaotic strange attractor (Arnold, V.I., 1984, 1988).

The geometric representation of the critical (ramification) points led René Thom, in 1972, to the notion of catastrophe, thus being developed the theory of catastrophes (Thom, R., 1972).
The evolution of the systems in time is modelled, obviously, by non-linear
differential equations, for which solutions in analytical closed form can be only very
seldom obtained. Even if these systems have been – at the beginning – only mechanical
ones, they involved afterwards all the chapters of physics, chemistry, biology etc. To
integrate these systems, one uses – in general – numerical algorithms; but all the
problems mentioned above are put. Thus appeared the theory of dynamical systems,
which has been very much developed last years; we can say that it is a qualitative
type of ordinary differential equations. Obviously, a dynamical system has a more
general significance than that of mechanical system, containing also electromagnetical,
biological, economical, social, political systems etc.

We present in this chapter continuous and discrete dynamical systems, for which we
put in evidence periodical solutions and global ramifications. We are thus led to
introduce some elements of catastrophe theory. In the theory of chaos (especially
deterministic chaos) we will use the notions of strange attractor, fractal etc.

24.1 Continuous and Discrete Dynamical Systems

The time has, in general, a continuous variation in the study of a dynamical system;
but one can consider also cases in which the results are obtained for discrete values of
the time variable. We will deal with both situations. As well, we make distinction
between linear and non-linear systems.

24.1.1 Continuous Linear Dynamical Systems

We introduce, in what follows, the most important notion necessary to the study of
dynamical systems, especially the notion of attractor. We mention also the study of
non-autonomous linear systems of differential equations with a control function or with
periodic coefficients.

24.1.1.1 Fixed Points. Attractors

Let be the non-autonomous system of differential equations with initial conditions
(of Cauchy type)

\[
\frac{dx}{dt} = X(x; t), \quad x(t_0) = x_0,
\]

(24.1.1)

where the column vectors

\[
x = [x_1 \ x_2 \ \ldots \ x_n], \quad X = [X_1 \ X_2 \ \ldots \ X_n]
\]

(24.1.1')