Appendix: Mathematical and Computational Tools

In the Appendix we give a brief review of some mathematical [II06c, II07e] and physiological concepts [II06b]—necessary for comprehensive reading of the book, followed by classical computational tools used in quantum neural computation.

7.1 Meta-Language of Categories and Functors

In modern mathematical sciences whenever one defines a new class of mathematical objects, one proceeds almost in the next breath to say what kinds of maps between objects will be considered [Swi75, II06c]. A general framework for dealing with situations where we have some objects and maps between objects, like sets and functions, vector spaces and linear operators, points in a space and paths between points, etc.—gives the modern metalanguage of categories and functors. Categories are mathematical universes and functors are ‘projectors’ from one universe onto another. For this reason, in this book we extensively use this associative meta-language, mainly following its founder, S. MacLane [Mac71].

7.1.1 Maps

Notes from Set Theory

Given a map (or, a function) \( f : A \to B \), the set \( A \) is called the domain of \( f \), and denoted \( \text{Dom } f \). The set \( B \) is called the codomain of \( f \), and denoted \( \text{Cod } f \). The codomain is not to be confused with the range of \( f(A) \), which is in general only a subset of \( B \).

A map \( f : X \to Y \) is called injective or 1–1 or an injection if for every \( y \) in the codomain \( Y \) there is at most one \( x \) in the domain \( X \) with \( f(x) = y \). Put another way, given \( x \) and \( x' \) in \( X \), if \( f(x) = f(x') \), then it follows that \( x = x' \). A map \( f : X \to Y \) is called surjective or onto or a surjection if for
every $y$ in the codomain $\text{Cod} f$ there is at least one $x$ in the domain $X$ with $f(x) = y$. Put another way, the range $f(X)$ is equal to the codomain $Y$. A map is bijective iff it is both injective and surjective. Injective functions are called the monomorphisms, and surjective functions are called the epimorphisms in the category of sets (see below).

Two main classes of maps (or, functions) that we will use in this book are: (i) continuous maps (denoted as $C^0$-class), and (ii) smooth or differentiable maps (denoted as $C^\infty$-class). The former class is the core of topology, the letter of differential geometry. They are both used in the core concept of manifold.

A relation is any subset of a Cartesian product (see below). By definition, an equivalence relation $\alpha$ on a set $X$ is a relation which is reflexive, symmetrical and transitive, i.e., relation that satisfies the following three conditions:

1. Reflexivity: each element $x \in X$ is equivalent to itself, i.e., $x \alpha x$,
2. Symmetry: for any two elements $a, b \in X$, $a \alpha b$ implies $b \alpha a$, and
3. Transitivity: $a \alpha b$ and $b \alpha c$ implies $a \alpha c$.

Similarly, a relation $\leq$ defines a partial order on a set $S$ if it has the following properties:

1. Reflexivity: $a \leq a$ for all $a \in S$,
2. Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$, and
3. Transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$.

A partially ordered set (or poset) is a set taken together with a partial order on it. Formally, a partially ordered set is defined as an ordered pair $P = (X, \leq)$, where $X$ is called the ground set of $P$ and $\leq$ is the partial order of $P$.

Notes From Calculus

Maps

Recall that a map (or, function) $f$ is a rule that assigns to each element $x$ in a set $A$ exactly one element, called $f(x)$, in a set $B$. A map could be thought of as a machine $[f]$ with $x$-input (the domain of $f$ is the set of all possible inputs) and $f(x)$-output (the range of $f$ is the set of all possible outputs) [Stu99]

$$x \rightarrow [f] \rightarrow f(x)$$

There are four possible ways to represent a function (or map): (i) verbally (by a description in words); (ii) numerically (by a table of values); (iii) visually (by a graph); and (iv) algebraically (by an explicit formula). The most common method for visualizing a function is its graph. If $f$ is a function with domain $A$, then its graph is the set of ordered input–output pairs

$$\{(x, f(x)) : x \in A\}.$$