Chapter 7
Solvers for Stochastic Galerkin Problems

The first chapters of this book were about the stochastic discretization of models involving some uncertain parameters. The focus was on the derivation of numerical or mathematical approaches for the definition and determination of the spectral coefficients for the stochastic solutions. In this chapter, we specifically focus on some numerical techniques and algorithms dedicated to the resolution of the mathematical problems arising from the stochastic Galerkin projection. These problems are characterized by their size, which is typically $(P + 1)$ times larger than their deterministic counterpart, where $(P + 1)$ is the dimension of the stochastic basis used. Since the deterministic problems of interest are often large and $P$ can be large as well, the resolution of the stochastic Galerkin problems is generally costly and requires efficient numerical strategies and appropriate numerical algorithms. In fact, even for simple linear problems, the assembly of the matrix representing the discrete system of equations to be solved is often impossible and its direct inversion is generally not an option.

In Sect. 7.1, we quickly review some Krylov methods for the resolution of large sparse linear problems. The methods discussed here are quite general and lead to iterative algorithms for the resolution of large systems of linear equations. Their application to systems arising from the Galerkin projection is discussed and a particular attention is given to the definition of appropriate preconditioners.

Section 7.2 describes a multigrid technique for the resolution of stochastic diffusion equations with random diffusivity. The multigrid algorithms rely on a coarsening of the spatial discretization to accelerate the convergence rate of the iterative solvers.

In Sect. 7.3 we detail Newton iterations for the resolution of the steady stochastic incompressible Navier-Stokes equations. This nonlinear solver combines Newton iterations with Krylov methods for non-symmetric problems discussed in Sect. 7.1, preconditioned by the linearized stochastic Navier-Stokes equations. This large nonlinear problem also illustrates the implementation of Krylov methods where the actual assembly of the system is never actually performed.
7.1 Krylov Methods for Linear Models

Linear models are of considerable importance as they represent a large variety of physical systems. In addition, the resolution of nonlinear models are generally based on series of corrections to the approximate solution, each consisting in the resolution of a linear problem such as in Newton-Raphson iterations. Therefore, solution techniques for linear systems of equations are a key ingredient of numerical algorithms and their efficiency, both in terms of computational time and memory requirement, are essential in high performance computing. This section discusses few iterative strategies for the resolution of large linear systems arising from the Galerkin projection of linear models. To recall notations and the structure of linear system of equations, we first briefly summarize the stochastic Galerkin projection procedure in the case of a linear model, which was detailed in Chap. 4 and fully exemplified in Chap. 5.

We assume that the stochastic problem has been already discretized at the deterministic level (using finite element, finite difference or any other technique), yielding a linear stochastic problem on the probability space \((\Xi, \mathcal{B}_\Xi, P_\Xi)\). This semi-discrete problem has the form

\[
A(\xi)u(\xi) = b(\xi), \quad \text{a.s.} \tag{7.1}
\]

where \(A(\xi) \in \mathbb{R}^{m \times m} \otimes L_2(\Xi, P_\Xi)\) is a stochastic matrix that will be assumed sparse, \(b(\xi) \in \mathbb{R}^m \otimes L_2(\Xi, P_\Xi)\) is the stochastic right-hand-side, and \(u(\xi) \in \mathbb{R}^m \otimes L_2(\Xi, P_\Xi)\) is the model solution. The stochastic discretization is performed by introducing a basis of orthogonal stochastic functionals spanning a finite dimensional stochastic subspace \(\mathcal{S}_p\) of \(L_2(\Xi, P_\Xi)\):

\[
\mathcal{S}_p = \text{span}\{\Psi_0, \ldots, \Psi_P\} \subset L_2(\Xi, P_\Xi), \quad \text{dim } \mathcal{S}_p = (P + 1). \tag{7.2}
\]

The developments in this section are independent of the stochastic basis, and we simply assume that by convention \(\Psi_0 = 1\), i.e. mode 0 corresponds to the mean mode. To lighten the notation we consider orthonormal bases:

\[
\langle \Psi_i, \Psi_j \rangle = \int_{\Xi} \Psi_i(y) \Psi_j(y) p_\Xi(y) \, dy = \delta_{i,j}, \quad \forall i, j = 0, \ldots, P. \tag{7.3}
\]

The stochastic expansion of the solution on \(\mathcal{S}_p\) is expressed as:

\[
u(\xi) \approx \sum_{k=0}^{P} u_k \Psi_k(\xi), \quad u_k \in \mathbb{R}^m, \quad k = 0, \ldots, P. \tag{7.4}
\]

Proceeding with the stochastic Galerkin projection, (7.1) becomes:

\[
\sum_{i=0}^{P} \langle A(\xi) \Psi_i, \Psi_k \rangle u_i = \langle b(\xi), \Psi_k \rangle = b_k, \quad k = 0, \ldots, P. \tag{7.5}
\]