Chapter 8
Wavelet and Multiresolution Analysis Schemes

An essential aspect of spectral representations analyzed so far is the projection of the process or solution on a polynomial basis, namely on a vector space spanned by infinitely differentiable functions. It is known following the works of Wiener [241] and Cameron and Martin [25] that such spectral representation converges in a mean square sense as $N, N_0 \to \infty$. In the case of numerical simulation, however, one must necessarily rely on truncation, and in this case the convergence may be seriously compromised due to truncation errors and aliasing, and in extreme cases these phenomena may lead to breakdown of the computations.

To illustrate potential difficulties, consider the case of a chemical system whose state at time $t$ is specified using the vector, $c(t)$, of the concentrations of individual species. Suppose also that the system evolves according to a mechanism in which one elementary reaction has an uncertain rate, modeled as a random variable $q(\theta)$ in an abstract probability space $(\Theta, \Sigma, P)$. In this case, we seek the stochastic solution $c(t, q(\theta))$, which involves a development along a single stochastic dimension, i.e. $N = 1, \xi = \{\xi\}$. Truncated to order $N_0$, the spectral expansion is expressed as:

$$c(t, q(\theta)) \approx u(t, \xi(\theta)) = \sum_{k=0}^{P=N_0} u_k(t) \Psi_k(\xi(\theta)).$$

One immediately notes that $u(t, \xi)$ is infinitely differentiable in $\xi$. Suppose now that the reaction mechanism admits a critical value of $q$, denoted by $q_c$, such that

$$\lim_{q \to q^-} c(t, q) \neq \lim_{q \to q^+} c(t, q),$$

i.e. the state $c$ exhibits a parametric bifurcation for $q = q_c$. One runs into such situations when ignition (or extinction) of one or more reactions occurs, due for instance to threshold effects. If $q_c$ is realizable, one effectively seeks to approximate, using infinitely differentiable functions, a functional admitting a bifurcation or a discontinuity. As for Fourier expansions where Gibbs phenomena are frequently illustrated, the lack of suitability of the spectral basis (too high of a regularity) leads to a very slow convergence in the case of infinite expansions, and to parasitic oscillations in
the finite case. Even for simple chemical systems [195], these effects can have catastrophic impact on the computations. Of course, these phenomena can also be present in other systems where bifurcations or steep variations with random inputs can occur. A well-known example concerns complex systems involving shock formation or an energy cascade [31], where PC expansions based on smooth polynomials can lose their advantages or cease to be useful.

In this chapter, we address difficulties arising when random data assume values in the neighborhood of a critical point (or more generally a critical surface), or exhibit steep variations. Attention is focused on particular situations where the solutions remain smooth and well behaved, but which can change dramatically, or even discontinuously, according to specific values of the uncertain data. Such sensitivity of the solution with regard to the random data can be viewed as a parameter shock or bifurcation. For reasons alluded to above, one may expect that Wiener-Hermite or other spectral expansions based on smooth functions would fail to adequately describe the steep (or discontinuous) dependence of the solution on the random data. This chapter explores the possibility of overcoming these difficulties by using a PC expansions based on Haar wavelets [47, 174, 234, 235] or multiwavelets [3]. In doing so, we combine concepts of generalized PC expansions (see Sect. 2.3), and of using of piecewise functions in stochastic Galerkin methods [50]. In contrast to global basis functions, wavelet and MW representations naturally lead to localized decompositions which suggest the possibility of a more robust behavior albeit at the expense of slower rate of convergence.

In Sect. 8.1, we focus our attention on the simplest family of wavelets, namely the Haar basis [47, 174, 234, 235]. We introduce the Wiener-Haar (WHa) expansion of solutions dependent on random data, and briefly outline the salient features of the resulting PC representation. Section 8.2 then illustrates applications of WHa expansions. We first address a simple model problem consisting of a dynamical system having two isolated, stable fixed points. Depending on specific realizations of the random initial conditions, the solution converges to one stable fixed point or the other. Numerical simulations indicate that the WHa scheme can effectively resolve the random process, but that a Wiener-Legendre (WLe) expansion is unsuitable in this case. A more complex situation is then considered, based on a stochastic version of the Rayleigh-Bénard problem. Specifically, we consider two cases involving a random Rayleigh number, with the uncertainty range either containing the critical point or lying entirely above the critical value.

In Sect. 8.3, we generalize the methodology of Sect. 8.1 to arbitrary polynomial order expansions according to the framework proposed by Alpert in [3], the WHa expansion corresponding to the zero order case. The goal of the present generalization is to explore the possibility of combining the advantages of a higher order convergence rate resulting from higher-order polynomial expansion ($p$-convergence), and of the robustness of local decompositions. A multiresolution analysis (MRA) scheme is specifically constructed, allowing for refinement of the expansion by increasing the number of resolution levels ($h$-refinement) and/or the polynomial order ($p$-refinement). In Sect. 8.4, the resulting scheme is tested based on application to the Lorentz system having a single random parameter. In particular, the convergence