In this chapter, we describe the basic questions related to properties and parametrization of rotations; the subject of the algorithmic treatment of rotations is addressed separately in Chap. 9.

The topic of finite rotations is very important in practice and often undertaken in the works on rigid-body dynamics, see [197, 79] and on multi-body dynamics of rigid and flexible bodies, see [258, 4, 44, 77, 76, 9]. There are also mathematical works on rotations, such as, e.g., [231, 45, 2]. This subject is also covered in the works on the Cosserat continuum and on structures with rotational degrees of freedom, such as shells and 3D beams; these works are cited in Chap. 4.

The rotations are described by a proper orthogonal tensor and its basic properties are presented in Sect. 8.1. However, in numerical implementations, we have to use some rotational parameters; several of them are in use and their properties are very different. We describe a wide, although not complete, selection of parametrizations in Sect. 8.2; some of them provide a theoretical background but are not used in computation of structures. For more details, see [5, 8, 107].

8.1 Basic properties of rotations

In this section, we provide elementary information related to rotations, such as the definition of the rotation tensor, and two basic problems: the rotation of a vector about an axis and the rotation of a Cartesian triad of vectors. Basic properties of orthogonal tensors and skew-symmetric tensors are provided.
8.1.1 Rotation tensor

Let us denote by \( R \) the rotation tensor belonging to the special orthogonal group defined as follows

\[
\text{SO}(3) := \{ R : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is linear} \mid R^T R = I \text{ and } \det R = +1 \}. \tag{8.1}
\]

The orthogonality condition \( R^T R = I \) renders that preserved are (i) the angle between two rotated vectors \( a, b \), because \( (Ra) \cdot (Rb) = a \cdot (R^T Rb) = a \cdot b \), and (ii) the length of a rotated vector, because \( \sqrt{(Ra) \cdot (Ra)} = \sqrt{a \cdot (R^T Ra)} = \sqrt{a \cdot a} \).

**Example.** The orthogonality condition \( R^T R = I \) itself does not suffice to define the rotation as the transformation representing a reflection also satisfies this condition. Consider two orthogonal matrices

\[
R_1 = \begin{bmatrix}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{bmatrix}, \quad R_2 = \begin{bmatrix}
\cos \omega & \sin \omega \\
\sin \omega & -\cos \omega
\end{bmatrix}. \tag{8.2}
\]

For the vector \( a = [1, 1]^T \) and \( \omega = \pi \), we obtain

\[
b = R_1 a = [-1, -1]^T, \quad c = R_2 a = [-1, 1]^T, \tag{8.3}
\]

which are shown in Fig. 8.1. Note that \( R_1 \) rotates \( a \) about point “0”, while \( R_2 \) reflects it across the line 0Y. We can check that \( \det R_1 = +1 \), while \( \det R_2 = -1 \).

![Fig. 8.1 Rotation of vector \( a \) yields vector \( b \); reflection yields vector \( c \).](image-url)