Notes on the Prime Number Theorem-I

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§ 1 Introduction and Notation

We begin by stating the Prime Number Theorem in a way somewhat different from the usual. Let $p_n$ denote the $n$-th prime (viz. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ...).

Then $p_n$ is given by the approximate formula

$$n = \int_2^{p_n} \frac{du}{\log u}$$

Of course this is only an approximation. The Prime Number Theorem in the usual form states that the error here is $O(p^{\frac{1}{2}} \log p_n)$ (on Riemann hypothesis namely $\zeta(s) \neq 0$ if $Re(s) > \frac{1}{2}$) and $O(p_n (\exp((\log p_n)^{\frac{3}{5}} (\log \log p_n)^{-\frac{1}{5}})))^{-\mu}$ (unconditionally) u.c., where $\mu > 0$ is an absolute constant, which is not very important ($O(\ldots)$ means a term whose absolute value is less than a constant times ...). Let $y \geq 2$ and $li y = \int_2^y \frac{du}{\log u}$. Then the equation $x = li y$ has a unique solution say $y = f(x)$ with $f(0) = 2$. From $x = li y$ it follows that $1 = (\log y)^{-1} \frac{dy}{dx}$ and hence $f'(x) = \log y$. From these it follows (by inverting the approximation for $p_n$ stated above), that

$$|p_n - f(n)| \leq \begin{cases} Cn^{\frac{1}{2}} (\log n)^{\frac{5}{2}}, & \text{on RH}, \\ CnE^{-D}, & \text{u.c.,} \end{cases} \quad (1)$$

where $C$ and $D$ are absolute positive constants and $E = \exp((\log n)^{\frac{3}{5}} (\log \log n)^{\frac{1}{5}})$. (Of course on the assumption $\zeta(s) = O((t^{(1-\sigma)n} (\log t))^A)(t \geq 2, \frac{1}{2} \leq \sigma \leq 1)$ where $a \geq 1$ and $A > 0$ are absolute constants there follow appropriate u.c forms of the error terms in PNT and the above things can be formulated in terms of such error terms also; the function $E$ above corresponds to $a = \frac{3}{2}$). Also a famous work of J.E. LITTLEWOOD states that $\pi(x)$ defined as $\sum_{p \leq x} 1$ satisfies

$$\pi(x) = li x + \Omega \left( \frac{x^{\frac{1}{2}}}{\log x} \log \log \log x \right),$$

this means that the upper and lower limits of $\pi(x) - li x$ divided by $\frac{x^{\frac{1}{2}}}{\log x} \log \log \log x$ as $x \to \infty$ are respectively positive and negative. From this it follows (on putting $x = p_n$ and
inverting) that
\[ p_n - f(n) = \Omega_{\pm}(n \log n)^{\frac{1}{2}} \log \log n \]
with a similar meaning (for \( \Omega_{\pm} \)) as explained just now. Thus on RH we have a very precise result for \( p_n - f(n) \) namely both \( O \) and \( \Omega_{\pm} \) results nearly of the same order of precision. What is usually customary is to state the results for \( \sum \rho^s \leq x \) \( \log \rho \) because this admits of a nice formula (called explicit formula) in terms of the zeros of \( \zeta(s) \). It should be mentioned that all the results on \( p_n \) stated above are equivalent to the corresponding \( (O \) and \( \Omega_{\pm} \)) results on \( \sum \rho^s \leq x \) \( \log \rho \), \( \pi(x) \) and so on.

The results of this paper are fairly well-known to experts working in this area. (It is meant for non-experts). The only justification for their publication is that we give a unified (and further generalised) version of the prime number theorem (on the number of primes \( \leq x \)) and Landau’s theorem (on the number of numbers \( \leq x \), which are either squares or sums of two squares) with the Vinogradov’s error term. (There are also other results such as Montgomery-Vaughan theorem on square free numbers). The first step (see §2) is that for any generalised Dirichlet series \( F(s) \) (defined by \( \sum_{n=1}^{\infty} a_n \lambda_n^{-s}, s = \sigma + it \)) satisfying some conditions we prove

\[
\frac{1}{2\pi i} \int_{C-iT}^{C+iT} F(s) \frac{x^s}{s} ds = \sum_{\lambda_n \leq x} a_n + E(T) \left( C = 1 + \frac{1}{\log x}, 2 \leq T \leq x \right) \tag{1*}
\]

where \( E(T) = O(xT^{-1} \exp((\log x)^\epsilon)) \) for every fixed \( \epsilon > 0 \) and the \( O \)-constant depends only on \( \epsilon \). (If \( f_1 \) is any complex number depending on some parameter and \( f_2 > 0 \) then \( f_1 = O(f_2) \) will mean that \( |f_1 f_2^{-1}| \) is bounded above. Some times we write \( O \ldots (f_2) \) to indicate that the bound depends on the constants . . . ). The next step (see §3) is to prove that \( F(s) \) is analytic in the rectangle \( (\sigma \geq 1 - \lambda(T), C_0 \leq |t| \leq T) \) for a constant \( C_0 > 0 \) and a suitable small positive function \( \lambda(T) \) and establish the bound

\[
F(s) = O(\exp((\log x)^\delta))
\]

on the boundary of this rectangle. The net result is

\[
\sum_{\lambda_n \leq x} a_n = M(x) + E_0(x, T)
\]

where \( E_0(x, T) = O(x(\exp((\log x)^{2\epsilon}))(T^{-1} + x^{-\lambda(T)})) \)

\[
M(x) = \frac{1}{2\pi i} \int_K F(s) \frac{x^s}{s} ds
\]

\( K \) being the contour obtained by joining \( 1 - \lambda(T) - iC_0, C - iC_0, C + iC_0 \) and \( 1 - \lambda(T) + iC_0 \) in this order by straight line segments. Choosing \( T \) by the requirement \( T = \exp(\lambda(T) \log x) \) we are led to our main theorem. (Note that \( M(x) \) can sometimes be so small as the error term itself, for example when \( F(s) = (\xi(s))^{-10} \)).

In our final choice of \( F(s) \), we can improve the error estimate in our main theorem by assuming Riemann hypothesis and its generalisations to \( L \)-functions involved in \( F(s) \). (See Remark 3 below). For unconditional results we have to depend on the deep estimate

\[
\left| \xi(s) - \frac{1}{s - 1} \right| \leq ((|t| + 10)^{(1-\sigma)^\eta} \log(|t| + 10))^{A}
\]

(4)