Chapter 4

Classification of Conics

4.1 Affine Classification of Conics

Convention: Let $A$ be a $2 \times 2$ matrix and $X$ be a point in $\mathbb{A}^2_R$. To make sense of the matrix multiplication $AX$, we consider $X$ as a column vector or a matrix of type $2 \times 1$. In other contexts, we shall adhere to the standard practice of viewing vectors in $\mathbb{R}^2$ as ordered pairs, row vectors or matrices of type $1 \times 2$. The inner product $\langle X, Y \rangle$ stands for the standard dot product in $\mathbb{R}^2$.

Let us recall the definition of an affine conic:

Definition 4.1.1. Let $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ be a $2 \times 2$ symmetric matrix of real numbers with at least one of the entries $a, b$ and $h \neq 0$ and $P = (g, f)$ be a point in the affine plane $\mathbb{A}^2_R$.

An affine conic $C$ in $\mathbb{A}^2_R$ is the set $C := \{X = (x, y) \in \mathbb{A}^2_R: \langle AX, X \rangle + \langle P, X \rangle + c = 0\}$ where $c$ is a fixed real number.

Let us note that $\langle AX, X \rangle = ax^2 + 2hxy + by^2$ and $\langle P, X \rangle = gx + fy$. Therefore $C = \{(x, y) \in \mathbb{A}^2_R : ax^2 + ahxy + by^2 + gx + fy + c = 0\}$. Our aim is to classify all the affine conics.

Theorem 4.1.2. Let $C$ be an affine conic defined by $C = \{(x, y) \in \mathbb{A}^2_R : ax^2 + 2hxy + by^2 + gx + fy + c = 0\}$.

Then $C$ is affinely equivalent to one of the following:

1. the circle $\{(x, y) \in \mathbb{A}^2_R : x^2 + y^2 = 1\}$,
2. the hyperbola $\{(x, y) \in \mathbb{A}^2_R : x^2 - y^2 = 1\}$,
3. the parabola \( \{(x, y) \in \mathbb{A}_{\mathbb{R}}^2 : y = x^2\} \),
4. the pair of intersecting lines \( \{(x, y) \in \mathbb{A}_{\mathbb{R}}^2 : xy = 0\} \),
5. two parallel lines \( \{(x, y) \in \mathbb{A}_{\mathbb{R}}^2 : x = \pm 1\} \),
6. coincidental lines,
7. singleton, or
8. the empty set.

**Proof.** We claim that we may assume that either \( a \neq 0 \) or \( b \neq 0 \). Suppose \( a = b = 0 \). Then the equation of the conic \( C \) is \( 2hxy + gx + fy + c = 0 \) with \( h \neq 0 \).

Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation defined by

\[
T(x, y) := (x + y, x - y) := (u, v)
\]

Then \( u + v = 2x \) and \( u - v = 2y \). Hence a point \( (x, y) \in C \) iff

\[
0 = 2h \frac{u + v}{2} \frac{u - v}{2} + g \frac{u + v}{2} + f \frac{u - v}{2} + c
= \frac{h}{2} u^2 - \frac{h}{2} v^2 + \frac{g + f}{2} u + \frac{g - f}{2} v + c.
\]

Therefore with respect to the new coordinates \((u, v)\) in \( \mathbb{R}^2 \), the conic \( C = \{(u, v) \in \mathbb{A}^2_{\mathbb{R}} : h u^2 - h v^2 + (g + f)u + (g - f)v + c = 0\} \). Hence we may assume that \( a \neq 0 \) or \( b \neq 0 \).

Let us assume that \( a > 0 \). Consider the affine transformation defined by

\[
\varphi(x, y) := (\sqrt{a} x, y) = (u, v).
\]

With respect to the new coordinates \((u, v)\) a point \( (x, y) \in \mathbb{A}^2_{\mathbb{R}} \) is written as \( (x, y) = (\frac{u}{\sqrt{a}}, v) \). Therefore a point \( (x, y) \in \mathbb{A}^2_{\mathbb{R}} \) is in \( C \) iff

\[
0 = ax^2 + 2hxy + by^2 + gx + fy + c
= a \left(\frac{u}{\sqrt{a}}\right)^2 + 2h \left(\frac{u}{\sqrt{a}}\right) v + bv^2 + g \frac{u}{\sqrt{a}} + f v + c
= u^2 + \frac{2h}{\sqrt{a}} uv + bv^2 + g \frac{u}{\sqrt{a}} + f v + c.
\]

Hence the conic \( C \) is affinely equivalent to the conic defined by an equation of the form \( x^2 + 2h'xy + by^2 + g'x + f'y + c = 0 \), and so we may assume that \( C := \{(x, y) \in \mathbb{A}_{\mathbb{R}}^2 : x^2 + 2hxy + by^2 + gx + fy + c = 0\} \).