Combinatorial Methods in Indian Music: Pratyayas in Saṅgītaratnākara of Sārṅgadeva

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1. Introduction

Six combinatorial tools (called pratyayas) have been in systematic use in India for the study of Sanskrit prosody (Chandas-śāstra) and these go back in time at least to Piṅgala (c. 300 BC?). Among these, three—prastāra (an enumeration rule for generating all the possible metrical patterns of a given class as a sequence of rows), uddiṣṭa (the process for finding, for any given metrical pattern, the corresponding row number in the prastāra) and naṣṭa (the converse of uddiṣṭa)—are found in Bharata’s Nāṭyaśāstra, in the chapter where prosody is discussed. Incidentally, the problem of placing Piṅgala and Bharata in the chronology of time still remains an unsettled question. The notion of pratyayas was perhaps discussed in other ancient texts of music also. However, the first extant text on music where the pratyayas are systematically dealt with, both in connection with patterns of musical phrases (tānas) and patterns of musical rhythms (tālas), is Saṅgītaratnākara of Sārṅgadeva (c.1225 AD). Nārāyaṇa Paṇḍita in his Gaṇitakaumudī (1356 AD) deals with some of these questions in a more general context, though his theory does not cover the kind of tāla-prastāra considered by Sārṅgadeva.

Our aim in this article is to highlight the contributions of Sārṅgadeva and explain his work in a mathematical set up. We first discuss the sequential generation or enumeration of patterns of musical phrases, called tāna-prastāra. The method of generating these patterns, as discussed in the first chapter of Saṅgītaratnākara, is essentially a rule for generating sequentially the $n!$ permutations of $n$ symbols. We note that the prastāra, and the naṣṭa and uddiṣṭa processes are all indeed encoded in a certain unique representation of any integer in terms of sums of factorials. We also explain how Sārṅgadeva employs a tabular figure, khaṇḍa-meru, to essentially go back and forth between any integer and its representation as a sum of factorials.

We then move on to discuss the pratyayas for patterns of musical rhythms, tāla-prastāra. This theory has been dealt with at length in the sixth chapter of
Saṅgītaratnākara. It is in fact a generalisation of the theory of pratyayas for moric metres or mātṛā-vṛttas, where the short syllable (laghu) is taken to be of one mātṛā (metrical time unit) and the long syllable (guru) is taken to be of two mātṛās. Saṅgītaratnākara considers musical rhythmic patterns (tālas) made up of druṭa (of one time unit), laghu, guru, pluta, which are of 2, 4 and 6 durations respectively, of that unit of druṭa.

It should be noted that the notion of pluta, viewed as a prolated vowel, considered to be of the same duration as three laghus, appears already in ancient Sanskrit prosody. In fact the notion of pluta occurs in the Rgveda only three times, though much more frequently in other Saṃhitās and Brāhmaṇas. In classical Sanskrit, a pure consonant sound called a vyanjana (without any vowel attached) is said to be of half a mātṛā. A laghu is said to be of one mātṛā, guru of two mātṛās and a pluta of three mātṛās respectively. Clearly, as it is clumsy to handle fractions in music, time units of 1, 2, 4 and 6 were perhaps introduced and called druṭa, laghu, guru and pluta respectively.

Saṅgītaratnākara first presents a systematic method of enumerating all the tālas of a given time duration in a prastāra, and follows this up with a complete mathematical theory of pratyayas which is a generalisation of the corresponding theory for moric metres. An interesting feature of tāla-prastāra is that the total number of patterns (the saṃkhyānka), if laid out in a sequence, has a generating function which involves a polynomial of the sixth-degree. This is an analogue of the notion of saṃkhyās for the moric metres, which are given by the so-called Fibonacci numbers (a result which is already present in the seventh century Prākṛta text on prosody, Vṛttajītisamuccaya of Virahānka), which have as their generating function the inverse of the quadratic polynomial \((1 - x - x^2)\). In fact the sequence of saṃkhyānkas associated with tāla-prastāras satisfies a more complex (four term) recurrence relation and has the inverse of the sixth degree polynomial \((1 - x - x^2 - x^4 - x^6)\) as its generating function. Based on this generating function, we can compute the number of tāla patterns in the prastāra, which have a given number of druṭas, laghus, gurus or putas. This is an analogue of the pratyaya known as ekadvayādi-lagakriya in Sanskrit prosody, and is discussed in Saṅgītaratnākara for the tāla-prastāra by introducing various tabular figures called merus, which are constructed on the basis of systematic recurrence relations which can be derived from the generating function mentioned above.

The discussion of pratyayas in prosody and music lead to the study of combinatorics related to three important ways of representing any non-negative integer, representations which are widely in use even today. While discussing the pratyayas for varṇa-vṛttas, Pingala gave the procedure for finding the binary representation of integers. Much later, Nārāyaṇa Paṇḍita, in his Gaṇitakaumudi

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