CHAPTER 5

Characteristic Classes and Chern-Weil Construction

In this chapter, we discuss the theory of characteristic classes. In the first part we present the axioms for Chern classes of a complex vector bundle, and then prove their existence and uniqueness. In the second part we introduce the notion of connection on a smooth vector bundle and curvature of connection, with related geometric concepts, including Riemannian or Levi-Civita connection and unitary connection. We then use Chern-Weil theory to construct Chern classes for a smooth complex vector bundle with a connection, as de Rham cohomology classes of the base space of the bundle, represented by invariant polynomials in the curvature of the connection. In the final section, we discuss Pontrjagin classes of a real vector bundle. The facts about fibre bundles which we use in the first part may be found in Chapter 1. Further details about these may be found in Steenrod [60] or Husemoller [34].

5.1. Chern classes

Recall from Hatcher [27], p. 140, p. 212, that the complex projective space $\mathbb{C}P^n$ is the quotient space of $\mathbb{C}^{n+1} - \{0\}$ obtained by factoring out scalar multiplication. This is a CW-complex with one $2k$-cell for each $0 \leq k \leq n$, and with no cell of odd dimension. The chain groups of $\mathbb{C}P^n$ with coefficients in a commutative ring $R$ with unit are $C_{2k} \cong R$ and $C_{2k+1} \cong 0$ for $k = 0, 1, \ldots, n$, and the boundary operators vanish. It follows that $H^{2k}(\mathbb{C}P^n; R) \cong R$, and $H^{2k+1}(\mathbb{C}P^n; R) \cong 0$ for $k = 0, 1, \ldots, n$.

The cup product turns $H^*(\mathbb{C}P^n; R)$ into a graded commutative ring. If $\alpha \in H^2(\mathbb{C}P^n; R)$ is a generator, then $\{1, \alpha, \alpha^2, \ldots, \alpha^n\}$ is a basis of the cohomology ring $H^*(\mathbb{C}P^n; R)$, and $\alpha^{n+1} = 0$. In other words, $H^*(\mathbb{C}P^n; R) \simeq R[\alpha]/(\alpha^{n+1})$, which is a truncated polynomial ring in $\alpha$, $\alpha^{n+1}$ being the principal ideal generated by $\alpha^{n+1}$. By a standard result of topology

$$H^*(\mathbb{C}P^\infty; R) = \lim_{n \to \infty} H^*(\mathbb{C}P^n; R) = R[\alpha],$$

and the natural inclusion $i_P : \mathbb{C}P^n \to \mathbb{C}P^\infty$ (the projectivization of the linear embedding $i : \mathbb{C}^{n+1} \to \mathbb{C}^\infty$) induces epimorphism

$$i_P^* : H^*(\mathbb{C}P^\infty; R) \to H^*(\mathbb{C}P^n; R),$$
where $i_p^*(\alpha^k) = \alpha^k$ for $k \leq n$, and $i_p^*(\alpha^k) = 0$ for $k > n$.

Let $\pi : E \longrightarrow X$ be a fibre bundle with fibre $F$. Then $H^*(E; R)$ is a $H^*(X; R)$-module where the module structure is given by $\alpha \cdot \beta = \pi^*(\alpha) \cup \beta$ for $\alpha \in H^*(X; R)$, $\beta \in H^*(E; R)$. Then the Leray-Hirsch Theorem says

**Theorem 5.1.1.** If $H^n(F; R)$ is a free $R$-module of finite rank for each $n$, then there exist cohomology classes $\beta_i \in H^*(E; R)$ such that $j^*(\beta_i)$ is a basis of $H^*(F; R)$ for each fibre $F$, where $j : F \hookrightarrow E$ is the inclusion. Moreover, the map

$$H^*(X; R) \otimes H^*(F; R) \rightarrow H^*(E; R)$$

given by $\sum_{k,i} \alpha_k \otimes j^*(\beta_i) \mapsto \sum_{k,i} \alpha_k \cdot \beta_i$ is an isomorphism.

Note that the isomorphism involves only the additive structure and module structure, and it is not a ring isomorphism. Thus $H^*(E; R)$ is a free $H^*(X; R)$-module with basis $\{\beta_i\}$. A proof of the theorem may be found in Hatcher [27], p. 432.

Let $\text{Vect}(X)$ denote the set of equivalence classes of complex vector bundles over $X$.

**Definition 5.1.2.** The Chern classes are functions

$$c_k : \text{Vect}(X) \longrightarrow H^{2k}(X; \mathbb{Z}), \; k \geq 0,$$

satisfying the following axioms.

(i) $c_0(E) = 1$, and $c_k(E) = 0$ if $k > \text{rk}(E)$.

(ii) Functoriality. $c_k(f^*E) = f^*(c_k(E))$ for a pull-back bundle $f^*E$, $f : Y \longrightarrow X$.

(iii) Whitney sum formula. If $c = 1 + c_1 + c_2 + \cdots$, then $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$. In other words,

$$c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1) \cdot c_j(E_2).$$

(iv) Normalization. For the universal line bundle $\gamma^1 \longrightarrow \mathbb{C}P^\infty$, $c_1(\gamma^1)$ is a pre-assigned generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

The class $c_k(E) \in H^{2k}(X; \mathbb{Z})$ is called the $k$-th Chern class of $E$, and $c(E) \in H^*(X; \mathbb{Z})$ the total Chern class of $E$.

The first three axioms are satisfied if $c_0 = 1$ and $c_k = 0$ for $k > 0$, or if $c_k$ is replaced by $n^k \cdot c_k$ for a fixed nonnegative integer $n$. The last axiom excludes these possibilities. Later we shall choose $c_1(\gamma^1)$ in a specific way, namely, as the Euler class of the underlying real bundle of $\gamma^1$.

**Existence of the Chern classes.**

First recall from §3.3 the notion of a projective bundle. Let $\pi : E \longrightarrow X$ be a complex $n$-plane bundle, and $\pi_P : P(E) \longrightarrow X$ be the projective bundle associated to $E$, whose fibre over $x \in X$ is the space of lines in the fibre $\pi^{-1}(x)$.