Parameterization of rotation

The effective description of rotations has led to the development of numerous parameterization techniques presenting various properties and advantages, as described in the following review papers [239, 240, 241, 242, 243, 244, 245]. Whether originating from geometric, algebraic, or matrix approaches, parameterization of rotation is most naturally categorized into two classes: vectorial and non-vectorial parameterizations. The former refers to parameterization in which a set of parameters, sometimes called rotational “quasi-coordinates,” define a geometric vector, whereas the latter cannot be cast in the form of a vector. These two types of parameterizations are sometimes denoted as invariant and non-invariant parameterization, respectively.

The Cartesian rotation vector, the Euler-Rodrigues parameters, or the Wiener-Milenković parameters all are examples of vectorial parameterizations. These are all characterized by a minimal set of three parameters, which behave as the Cartesian components of a vector in three-dimensional space. Non-vectorial parameterizations, on the other hand, may be either minimal, as in the case of Euler angles, or “redundant,” as for Euler parameters, Cayley-Klein parameters, and the matrix of direction cosines.

Redundancy arises when more than three parameters are employed: four in the case of Euler and Cayley-Klein parameters, nine in the case of direction cosines. In fact, rotation may be described as the motion of a point on a three-dimensional nonlinear manifold, the Lie group of special orthogonal transformations of the three-dimensional space. The various parameterizations of rotation are, in differential geometry terminology, different charts available for this particular manifold.

Stuelpnagel [246] provides a concise analysis of different parameterizations of rotations. He shows that the six parameter representation consisting of the first two columns of the rotation tensor yields a set of linear differential equations for the motion of a rigid body. He further proves that a minimum of five parameters is required to obtain a bijective mapping of the rotation group. This parameterization yields a set of nonlinear equations of motion for a rigid body and is not recommended for practical applications. Four parameter representations, such as the quaternion representation [247, 248, 249], are singularity free, in contrast with minimal set parameterizations, which he proves to always involve singularities.
The various parameterization techniques detailed in the literature present distinct advantages and drawbacks. Advantages can be of a theoretical nature, such as ease of geometric interpretation, or convenience in algebraic manipulations, for instance, or of a computational nature, low cost function evaluations, wide range of singularity-free behavior, etc. These features provide guidelines for selecting parameterizations that are best suited for specific applications. A survey of the literature reveals, however, that for both theoretical and numerical applications, the choice of parameterization is often based on personal taste and traditions rather than cost/benefit considerations.

Section 13.1 presents an algebraic description of rotation that contrasts with the geometric approach developed in chapter 4. Cayley’s elegant formulation is introduced based on the fundamental property of the rotation operation: preservation the length of the rotated vector. Next, section 13.3 introduces the well-known Euler parameters [247, 246, 250, 249] that provide an elegant, purely algebraic representation of rotation. When using the quaternion algebra presented in section 13.2, all rotation operations become bi-linear expressions of quaternions. These advantages, however, come at a high cost: four parameters must be used instead of three, i.e., Euler parameters do not form a minimum set.

Euler’s theorem on rotations, see section 4.5, states that an arbitrary motion of a rigid body that leaves one of its point fixed can be represented by a single rotation of magnitude $\phi$ about unit vector $\vec{n}$. It is readily shown that the associated rotation tensor, $\mathbf{R}$, possesses a positive unit eigenvalue and the corresponding eigenvector is $\vec{n}$, see section 4.7.

The vectorial parameterization of rotation is introduced in section 13.4 and consists of minimal set of parameters defining the components of a rotation parameter vector, $p = p(\phi)\vec{n}$, where $p(\phi)$ is the generating function. The vectorial parameterization of rotation presents two fundamental properties. First, it is tensorial in nature: the tensorial nature of the second-order rotation tensor implies and is implied by the tensorial nature of the rotation parameter vector, a first-order tensor. Second, rotation parameter vectors are parallel to the eigenvector of the rotation tensor corresponding to its unit eigenvalue. Because these two properties imply each other, either can be taken as the definition of the vectorial parameterization of rotation. A parameterization of rotation is tensorial if and only if the rotation parameter vector is parallel to the eigenvector of the rotation tensor associated with its unit eigenvalue.

The Cartesian rotation vector, the Cayley-Gibbs-Rodrigues parameters, or the Wiener-Milenković parameters all are special cases of the vectorial parameterization of rotation corresponding to specific choices of the generating function. Furthermore, these parameterizations are recovered as members of two different families: the sine and the tangent family. The occurrence of singularities in the proposed vectorial parameterization is the focus of section 13.6. Finally, section 13.7 details a number of useful parameterizations: the Cartesian rotation vector, the Euler-Rodrigues parameters, the Cayley-Gibbs-Rodrigues parameters, and the Wiener-Milenković parameters.

Euler parameters are closely related to the vectorial parameterization. On the other hand, minimal non-vectorial parameterizations such as Euler and Euler-type