Multibody systems are characterized by two distinguishing features: system components undergo finite relative rotations and these components are connected by mechanical joints that impose restrictions on their relative motion. Broadly speaking, multibody systems can be divided into three categories, rigid multibody systems, linearly elastic multibody systems, and nonlinearly elastic multibody systems. This classification and its implication on modeling techniques for multibody systems are discussed in section 15.1.

Section 15.2 presents a review of the basic equations of three-dimensional, linear elastodynamics. Geometrically nonlinear problems are characterized by nonlinear strain-displacement relationships, which are the subject of section 15.3. In section 15.5, special attention is devoted to the formulation of problems where structures undergo arbitrarily large displacements and rotations although strain components are assumed to remain small at all points of the structure.

15.1 Classification of multibody systems

Multibody systems can be divided into three categories, rigid multibody systems, linearly elastic multibody systems, and nonlinearly elastic multibody systems. Systems of the first category involve rigid bodies only, but those of the latter two categories comprise both rigid and flexible bodies. Section 12.5.1 introduced the concept of floating frame of reference in which the total motion of flexible bodies is broken into two parts: rigid body motions represented by the motion of the floating frame of reference and superimposed elastic motions. By definition, rigid body motions generate no strains. The elastic motions typically consist of displacement and rotation fields, which generate an associated strain field, denoted $\varepsilon$. For rigid multibody system, the strain field vanishes in all bodies, i.e., $\varepsilon = 0$ in each body. The distinction between linearly and nonlinearly elastic multibody systems stems from the characteristics of the strain field.

The characteristics of the three types of multibody systems are as follows.
1. **Rigid multibody systems** consist of an assemblage of rigid bodies connected together through mechanical joints and in arbitrary motion with respect to each other. Although all bodies are rigid, i.e., $\xi = 0$ in each body, lumped elastic components, also called *flexible joints*, *bushing elements* or *force elements*, could be placed between two components of the system to represent localized elasticity. These flexible joints exhibit arbitrary constitutive behavior.

2. **Linearly elastic multibody systems** consist of an assemblage of both elastic and rigid bodies connected together through mechanical joints and in arbitrary motion with respect to each other. For linearly elastic multibody systems, it is assumed that the strain-displacement relationships remain linear and that strains components remain very small at all times, i.e., $\xi \ll 1$ for all elastic bodies. Efficient analysis techniques for this type of problems typically rely on modal expansions of the elastic displacement field.

3. **Nonlinearly elastic multibody systems** consist of an assemblage of both elastic and rigid bodies connected together through mechanical joints and in arbitrary motion with respect to each other. For the elastic bodies, the strain-displacement relationships become nonlinear, or the strain components become large, or both. Nonlinear strain-displacement relationships characterize *geometrically nonlinear problems*, i.e., problems involving large elastic displacements, or rotations, or both. When strain components become large, nonlinear material constitutive laws must be used, a characteristic of *materially nonlinear problems*. For nonlinearly elastic multibody systems the accuracy and reliability of modal expansion of the elastic displacement field become questionable.

Because the overall motions of all bodies of a multibody system are large and because the relative motions between the system’s various components are also large, multibody system dynamics is an inherently nonlinear problem. The qualifiers “linearly” and “nonlinearly elastic” used in the classification above specifically refer to the elastic behavior of the bodies. The modeling of linearly elastic multibody systems leads to nonlinear dynamical equations of motion, although the representation of the elastic behavior of the bodies could be largely linearized.

### 15.1.1 Linearly and nonlinearly elastic multibody systems

The demarcation between linearly and nonlinearly elastic multibody systems is sometimes blurry. Consider, for instance, the problem of a helicopter rotor blade. As the blade rotates, elastic displacements and rotations remain very small, and the blade is designed to undergo small strains at all time to ensure safety of flight and guarantee structural fatigue life. This problem seems to fall into the category of linearly elastic multibody systems.

Due to the high angular speed of the rotor, however, large centrifugal forces appear in the blade, leading to considerable centrifugal stiffening of the blade and nonlinear coupling between its two bending and torsional deformations. To accurately capture these effects, nonlinear strain-displacement relationships must be used, although linear constitutive laws adequately represent material behavior. These geo-