This chapter investigates applications of the principles of analytical mechanics developed in chapter 7 to dynamical systems. First, the principle of virtual work presented in section 7.5 for static problems will be generalized to dynamic problems, leading to d’Alembert’s principle, see section 8.1. Next, Hamilton’s principle is presented in section 8.2 as an integral version of d’Alembert’s principle. Finally, Lagrange’s formulation is presented in section 8.3, leading to Lagrange’s equations of motion.

8.1 D’Alembert’s principle

Newton’s second law, eq. (3.4), states that if external forces, \( F^a \), are acting on a particle, its acceleration is proportional to the sum of these forces, \( F^a = ma \). The product of the mass by the acceleration vector is a force vector, called the inertial force vector, \( F^I \), defined as

\[
F^I = -ma.
\]

The minus sign in the definition of the inertial force indicates that such force always opposes motion. With this definition, Newton’s second law becomes

\[
F^I + F^a = 0.
\]

Of course, this equation looks like a trivial manipulation of Newton’s law: inertial forces have been brought from the right- to the left-hand side of the equation. The importance of the above statement, however, is that it generalizes the concept of equilibrium, a concept of statics, to dynamics problems.

As mentioned in section 3.1.2, Newton’s first law is generally stated as “a particle is in static equilibrium if and only if the sum of the externally applied forces vanishes” within the context of statics problems.

Equation (8.2) expresses the condition for dynamic equilibrium: the sum of the externally applied forces must vanish, provided that the inertial forces are treated as externally applied forces. Of course, the concept of dynamic equilibrium does not imply that the particle is at rest; indeed, the particle moves under the effect of...
the externally applied forces. Rather, dynamic equilibrium implies the vanish of the resultant of the set of forces acting on a particle in motion; this set of forces includes all externally applied forces and the inertial forces. The importance of the concept of inertial force is that the same law, “the sum of the forces must vanish,” now applies to both statics and dynamics problems; dynamics is reduced to statics. D’Alembert’s principle can now be stated as follows.

**Principle 12 (D’Alembert’s principle)** A system of particles is in dynamic equilibrium if and only if the sum of the externally applied forces and inertial forces vanishes.

In section 7.5, the principle of virtual work for static problems was derived from Newton’s first law and shown to imply that \( \delta(V) = \delta W_{nc} \), for all arbitrary virtual displacements, see eq. (7.40). In this expression, \( V \) is the potential of the conservative forces acting on the system of particles, and \( \delta W_{nc} \) the virtual work done by the non-conservative forces.

For dynamic equilibrium, D’Alembert’s principle requires the vanishing of the sum of the externally applied forces and inertial forces. Inertial forces are non-conservative force because they cannot be derived from a potential. It follows that the principle of virtual work, the condition for static equilibrium, can be generalized to becomes the condition for dynamic equilibrium, if the virtual work done by the inertial forces, denoted \( \delta W^I \), is added to the virtual work done by the other non-conservative forces. In summary, a system of particles is in dynamic equilibrium if and only if

\[
\delta(V) = \delta W_{nc} + \delta W^I,
\]

for all arbitrary virtual displacements. D’Alembert’s principle can also be stated as follows.

**Principle 13 (D’Alembert’s principle)** A system of particles is in dynamic equilibrium if and only if virtual changes in the potential of the conservative force equal the virtual work done by the non-conservative forces and inertial forces for all arbitrary virtual displacements.

The principle of virtual work presented in section 7.5 is equivalent to Newton’s first law. By treating inertial forces as “externally applied forces,” dynamic problems are reduced to static problems and d’Alembert’s principle becomes equivalent to Newton’s second law. The two alternative statements of d’Alembert’s principle given above are equivalent to Newton’s second law, and hence, provide an alternative basis for dynamics.

For a system composed of \( N \) of particles, the virtual work done by the inertial forces is

\[
\delta W^I = \sum_{i=1}^{N} F_i^T \delta \mathbf{r}_i = -\sum_{i=1}^{N} m_i a_i^T \delta \mathbf{r}_i,
\]

where \( a_i \) is the inertial acceleration vector of the \( i^{th} \) particle, \( m_i \) its mass, and \( \delta \mathbf{r}_i \) an arbitrary virtual displacement of the same particle.