Chapter 9
Block Towers: Co-construction of Proof

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Tasks: Towers
Participants: Ali, Angela, Magda, Michelle, Robert, and Sherly
Researchers: Carolyn Maher, Alice Alston, Susan Pirie

9.1 Introduction

In previous chapters, we observed elementary school students working to make sense of the towers problems by building representations, formulating conjectures, and defending their solutions in discussions with classmates and researchers. In this chapter, we observe a cohort of high school juniors as they engage in explorations and constructions in the towers problem. During this session, the students found and generalized formulas for solutions to the original towers problem (building towers when selecting from two colors of Unifix cubes) and extensions (with more than two colors of cubes), using methods including controlling for variables, justification by cases, and inductive reasoning.

9.2 Building Towers

In the 2-h session, students worked in pairs on tower problems. They came up with a general rule for the number of possible towers of height $n$ when selecting from $x$ colors ($x^n$) and an explanation of that result based on an inductive argument based on generating all possible towers of a given height. Their arguments contained reasoning by cases, induction, and reasoning by contradiction. In addition, Robert produced an equation for the number of towers having exactly two cubes of one color (when selecting from two colors), for a tower of any height.
### 9.2.1 Angela and Magda

Neither Angela nor Magda had previous experience with the towers problem, as they had both joined the longitudinal study in sixth grade. In this, their first experience with towers, they found all 16 towers, four-tall, selecting from two colors. Interestingly, they used strategies similar to those developed by the fourth and fifth graders that participated earlier in the study (see Chapter 4). The girls organized their work by cases: (1) one blue, (2) two blues, (3) three blues, and (4) four of the same color. The two single-color cases consisted of one tower each; the one-blue-cube and three-blue-cube cases exhibited a local organization; they built those towers by moving the single cube of one color into each of the four possible positions. When asked how they knew that they had all the towers with one blue cube, they described their organization:

**MAGDA:** The blue is in each position each time.

**ANGELA:** Yeah, each possible position because there’s only four spots.

Initially, they had no support for accounting for the towers in the two-blue-cube case; they explained that they were unable to find any more. However, after they found four of the towers for this case, and they were asked how they knew they had them all, Angela alluded to a preliminary organization using controlling for variables strategy, that is, holding the top and bottom cubes constant. She said:

*Well, I mean, I don’t know how to explain it, there’s just like no other possibilities for it. I mean, there’s only four places, you have them, like you know, yellow on top, blue on the bottom, and the blue on top, yellow on bottom, then blue on top and bottom, and yellow on top and bottom.*

As she was saying this, Angela found the two towers for this case that were missed. There are two towers with yellow on the top and blue on the bottom and two towers with blue on the top and yellow on the bottom; they had originally found only one of each of those pairs.

When asked to determine the number of three-tall towers, Angela and Magda moved from building towers to drawing them, again with an organization by cases. The eight towers that they found were organized in three cases: (1) one blue cube, (2) two blue cubes, and (3) all one color. Each case was locally organized, as shown in Fig. 9.1. After thinking about their findings, they developed a general rule; according to what they called “Angela’s Law of Towers,” the number of \( n \)-tall towers when you have \( x \) colors to choose from is \( x^n \). Thus Angela and Magda not only provided a solution to the specific four-tall towers problem posed, but they also posed a generalization from towers with two colors to towers with \( x \) colors.

![Case 1](BYY)
![Case 2](BBY)
![Case 3](BY)

**Fig. 9.1** Angela and Magda’s list of three-tall towers