Chapter 2

Thermal conduction in a half-strip and a sector

1. Asymptotic expansion for the thermal conduction in a plate

A plate is a three-dimensional body, a dimension of which is thinner than the other ones. That dimension is denoted thickness. If we want to use a finite elements method to compute the solution of a mechanical problem posed over a plate, it will cost a lot in memory. If we take, for instance, ten elements in the thickness, we have to take several thousands in the length and the mesh becomes quickly huge.

Geometrical considerations based on the small thickness lead to the study of the plate as a two-dimensional body composed by the middle plane. These considerations are not based on the mechanical properties but only on geometrical hypothesis. Many authors have tried to justify this model. We can mention A. L. Gol'denveizer [G], who built an asymptotic expansion of the three-dimensional solution with respect to the thickness of the plate. Then P. G. Ciarlet and P. Destuynder [CD1], [CD2], P. Destuynder [D1], [D2] have proved the convergence of the three-dimensional solution to the two-dimensional solution as the thickness approaches zero. D. Caillerie [C] has obtained similar results with non-homogeneous plates with periodical structure. This asymptotic expansion method allows us to find the same approximation as geometrical considerations.

The temperature and thermal flux for the thermal conduction problem, established by the asymptotic expansion are good approximations. But close to the lateral surface, they are not suited. For a laminated plate (a plate which is formed by the track of several materials), the damage phenomena (crack, delamination,...) appear on the edge, at the interfaces between two different materials. It is why, to study and predict the damage phenomena, we need a good approximation of the thermal flux close to the edge. Therefore, we must improve the description obtained by the asymptotic expansion. To do this, we use the boundary layer theory.
(K. O. Friedrichs and R. F. Dressler [FD], E. Sanchez-Palencia [S], H. Dumontet [D]). In the vicinity of the edge, a corrective term is added to the leading term of the asymptotic expansion of the thermal conduction. Therefore, we obtain a local description. Thus, the leading term of our expansion is expressed as the sum of the stress or thermal flux suited far from the edge and a corrective term, a boundary layer term.

Nevertheless, the structure of this boundary layer term is not known: the behavior at infinity (far from the edge) of the corrective term must not modify the classical asymptotic expansion. In order to get the description of the boundary layer term, we shall express it as a function of root functions of some operator. To do this, we have to prove the completeness of a system of root functions for the problem of steady thermal conduction.

1.1. THE STEADY THERMAL CONDUCTION PROBLEM IN A PLATE

Let \( \omega \) be a domain of \( \mathbb{R}^2 \) with coordinates \((X_1, X_2)\). The boundary of \( \omega \) is denoted by \( \partial \omega \). Given a constant \( \varepsilon > 0 \), we define the plate \( \Omega^\varepsilon \), its upper and lower faces \( \Sigma^{+\varepsilon} \) and \( \Sigma^{-\varepsilon} \), respectively, and the lateral surface \( \Gamma^\varepsilon \) by:

\[
\begin{align*}
\Omega^\varepsilon &:= \omega \times (-\varepsilon, \varepsilon), \\
\Gamma^\varepsilon &:= \partial \omega \times (-\varepsilon, \varepsilon), \\
\Sigma^{+\varepsilon} &:= \omega \times \{\varepsilon\}, \\
\Sigma^{-\varepsilon} &:= \omega \times \{-\varepsilon\},
\end{align*}
\]

where \( \varepsilon \) represents the ratio between the thickness and the length of the middle plane of the plate.

![Figure 3. The plate \( \Omega^\varepsilon \)](image)

The outer normal to the boundary of \( \Omega^\varepsilon \) will be denoted by \( \mathbf{n} = (n_1, n_2, n_3) \) (a boldface letter will denote a vector.)

In this part, the problem we are interested in, is the steady thermal conduction problem. We shall remind some classical notations used in the framework. The main unknown of the problem is the temperature field \( u^\varepsilon (X) \), where \( X = (X_1, X_2, X_3) \), and \( X_3 \) is the thickness axis. The thermal flux vector field