12 Controllability and Observability

12.1 Introduction

Controllability measures the ability of a particular actuator configuration to control all the states of the system; conversely, observability measures the ability of the particular sensor configuration to supply all the information necessary to estimate all the states of the system. Classically, control theory offers controllability and observability tests which are based on the rank deficiency of the controllability and observability matrices: The system is controllable if the controllability matrix is full rank, and observable if the observability matrix is full rank. This answer is often not enough for practical engineering problems where we need more quantitative information. Consider for example a simply supported uniform beam; the mode shapes are given by (2.55). If the structure is subject to a point force acting at the center of the beam, it is obvious that the modes of even orders are not controllable because they have a nodal point at the center. Similarly, a displacement sensor will be insensitive to the modes having a nodal point where it is located. According to the rank tests, as soon as the actuator or the sensor are slightly moved away from the nodal point, the rank deficiency disappears, indicating that the corresponding mode becomes controllable or observable. This is too good to be true, and any attempt to control a mode with an actuator located close to a nodal point would inevitably lead to difficulties, because this mode is only weakly controllable or observable. In this chapter, after having discussed the basic concepts, we shall turn our attention to the quantitative measures of controllability and observability, and apply the concept to model reduction.

12.1.1 Definitions

Consider the linear time-invariant system

\[ \dot{x} = Ax + Bu \]  

(12.1)
\[ y = Cx \] (12.2)

- The system is \textit{completely controllable} if the state of the system can be transferred from zero to any final state \( x^* \) within a finite time.
- The system is \textit{stabilizable} if all the unstable eigenvalues are controllable or, in other words, if the non-controllable subspace is stable.
- The system is \textit{completely observable} if the state \( x \) can be determined from the knowledge of \( u \) and \( y \) over a finite time segment. In the specialized literature, observability refers to the determination of the current state from future output, while the determination of the state from past output is called reconstructibility. For linear, time-invariant systems, these concepts are equivalent and do not have to be distinguished.
- The system is \textit{detectable} if all the unstable eigenvalues are observable, or equivalently, if the unobservable subspace is stable.

### 12.2 Controllability and Observability Matrices

The simplest way to introduce the controllability matrix is to consider the single input \( n \)-dimensional discrete-time system governed by the difference equation

\[ x_{k+1} = Ax_k + bu_k \] (12.3)

where \( A \) is the \( n \times n \) system matrix and \( b \) the \( n \)-dimensional input vector. Assuming that the system starts from rest, \( x_0 = 0 \), the successive values of the state vector resulting from the scalar input \( u_k \) are

\[
egin{align*}
x_1 &= bu_0 \\
x_2 &= Ax_1 + bu_1 = Abu_0 + bu_1 \\
\vdots \\
x_n &= A^{n-1}bu_0 + A^{n-2}bu_1 + \ldots + bu_{n-1}
\end{align*}
\]

or

\[
x_n = (b, Ab, A^2b, \ldots, A^{n-1}b) \begin{pmatrix} u_{n-1} \\ \vdots \\ u_1 \\ u_0 \end{pmatrix}
\] (12.4)

where \( n \) is equal to the order of the system. The \( n \times n \) matrix

\[
C = (b, Ab, A^2b, \ldots, A^{n-1}b)
\] (12.5)

is called the \textit{controllability matrix}; its columns span the state space which can be reached after exactly \( n \) samples. If \( C \) is full rank, the state vector can be transferred to any final value \( x^* \) after only \( n \) samples. By solving Equ.(12.4), one finds