10.1 TRANSPORTATION PROBLEM WITH LINEAR FRACTIONAL OBJECTIVE FUNCTION

In this section we deal with the following transportation problem with linear fractional objective function: Given two cost matrices \( C = (c_{ij}) \) and \( D = (d_{ij}), i = 1, \ldots, m; j = 1, \ldots, n \), determine a matrix \( X = (x_{ij}) \) which minimizes the function:

\[
Z(X) = \frac{N(X)}{D(X)} = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + c_0 \right) / \left( \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} + d_0 \right)
\]

with the constraints

\[
\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m,
\]

\[
\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n,
\]

\[
x_{ij} \geq 0 \text{ for all } i \text{ and } j
\]

where \( c_0 \) and \( d_0 \) are constants.

Denote by \( S \) the domain of the feasible solutions of the problem (10.1.1)-(10.1.4).

Assume

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} + d_0 > 0 \text{ for all } X \in S,
\]

\[
\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j
\]

This last condition expresses the fact that the problem is balanced and constitutes the necessary and sufficient condition for the problem to have a solution. The optimal solution appears in a vertex of \( S \) and the local optimum is global.

As in the linear case, the solution of the fractional transportation problem takes place in two steps:

1. The determination of a initial solution,
2. The determination of the optimal solution.
The determination of the initial solution can be done using either the method of the North-West corner, or the method of the minimum element on the row (column or of the table), or Vogel's method (the procedure of maximum differences).

The second step consists in the verification of the solution optimality (starting with the initial solution) and, if it is not optimal, in its improvement.

Let \( I = \{1, \ldots, m\} \times \{1, \ldots, n\} \) and \( X = \{(x_{ij}) \mid (i, j) \in I \} \) be a basic feasible solution. Denote by \( I_x = \{(i, j) \in I \mid x_{ij} > 0, \ x_{ij} \in X \} \) the pairs of indices of the base variables.

Due to the condition \( (10.1.5') \) each non-degenerate basic solution will contain \( m+n-1 \) positive components.

Consider \( u_i^1, \ v_j^1 \) (\( u_i^2, \ v_j^2 \), respectively) the simplex multipliers (or dual variables) associated with the function \( N \) \((D, \text{respectively})\) defined such that to have

\[
\begin{align*}
  u_i^1 + v_j^1 &= c_{ij} \quad (i, j) \in I_x \\
  u_i^2 + v_j^2 &= d_{ij} \quad (i, j) \in I - I_x
\end{align*}
\]

For the variables outside the base, consider the comparative costs

\[
\begin{align*}
  \tilde{c}_{ij} &= c_{ij} - u_i^1 - v_j^1, \\
  \tilde{d}_{ij} &= d_{ij} - u_i^2 - v_j^2 \quad (i, j) \in I - I_x
\end{align*}
\]

The systems \((10.1.6)\) and \((10.1.7)\) can be solved independently, each having \( m + n - 1 \) equations and \( m + n \) unknowns. Their solution is obtained giving the value zero to one unknown. Being given the feasible solution \( X \), consider another solution \( X^* \) which differs from \( X \) by bringing into the base the variable \( X_{pq} \). We have the following relations

\[
N(X^*) = N(X) + \theta \tilde{c}_{pq} \text{ and } D(X^*) = D(X) + \theta \tilde{d}_{pq}, \text{ where } \theta \text{ is a value attributed to the variable } X_{pq}.
\]

Since,

\[
Z(X^*) - Z(X) = (N(X) + \theta \tilde{c}_{pq})/(D(X) + \theta \tilde{d}_{pq}) - N(X)/D(X) = \theta[\tilde{c}_{pq} D(X) - \tilde{d}_{pq} N(X)]/[D(X)(D(X) + \theta \tilde{d}_{pq})]
\]

it follows that \( X^* \) is a better solution than \( X \) if

\[
\Delta_{pq} = \tilde{c}_{pq} D(X) - \tilde{d}_{pq} N(X) < 0.
\]

This yields the optimality criterion for a solution.

**Theorem 10.1.1.** A feasible solution \( X = (x_{ij}) \) is a local optimum of the problem \((10.1.1)-(10.1.4)\) if

\[
\Delta_{pq} = \tilde{c}_{pq} D(X) - \tilde{d}_{pq} N(X) \geq 0, \quad (p, q) \in I - I_x.
\]

For the variables which belong to the base we have \( \Delta_{pq} = 0 \). If for the variables from outside the base there are negative values \( \Delta_{pq} \), then \( \Delta_{ij} = \min \{\Delta_{pq} \mid \Delta_{pq} < 0\} \) is determined and the variable \( x_{ij} \) is introduced in the base. Otherwise, one proceeds as in the usual transportation problem.